Ay 121: Homework 5

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[1] Dispersion Relation for magnetized plasma. In the class I derived the dispersion relation for propagation of EM waves in unmagnetized dilute and cold ($k_B T \ll m_e c^2$). Below, the k-vector is along the y axis and the external magnetic field, $\vec{B}_0$ is along the z. Now, using the framework as presented in the class, derive the four dispersion relations for propagation in cold plasma: (1) $\vec{k} \perp \vec{B}_0$ (linearly polarized signal, $\vec{E} \parallel \vec{B}_0$; linearly polarized signal, $\vec{E} \perp \vec{B}_0$); (2) $\vec{k} \parallel \vec{B}_0$ (separately for each sense of circular polarization) [Ignore contribution to the plasma frequency from ions]. [$4 \times 7.5 = 30$ points]

[2] Ground-level energy of helium. Let E_0 be the binding energy for the helium – that is the minimum energy needed to strip away both electrons. The first ionization potential of helium is measured to be 24.587 eV. Just given this information and with what you learnt in undergrad physics compute E_0 in eV and in Rydberg. [5 points]

Now the main problem. The formal non-relativistic Hamiltonian for helium is

$$H = -\frac{\hbar^2}{2m_e}(\nabla_1^2 + \nabla_2^2) - \frac{Ze^2}{r_1} - \frac{Ze^2}{r_2} + \frac{e^2}{r_{12}} \tag{1}$$

with $Z = 2$. Let us set $H(0)$ equal to the first three terms of the RHS and H' to the last term of the RHS. The spatial part of the ground-state eigenfunction of $H(0)$ is $\psi_{\mathrm{gs}(\mathrm{r_1},\mathrm{r_2})} = \phi_{\mathrm{1s},Z}(r_1)\phi_{\mathrm{1s},Z}(r_2)$ where $\phi_{1s}(r)$ is the 1s normalized wave-function of the hydrogen atom but for $Z = 2$. You can readily conclude that the eigenvalue for $H(0)$ is $E_{\mathrm{gs}}^{(0)} = -2Z^2$ Rydberg (Ryd).

We adopt $\psi_{\mathrm{gs}}(r_1, r_2)$ and then compute $E' = \langle H' \rangle$ where $H' = e^2/r_{12}$ and the angular brackets stand for $\int\int \psi H' \psi^* d^3r_1 d^3r_2$. Analytically[1] show that $E' = {}^5\!/\!_4 Z$ Ryd. Thus, for

[1] The integral can be done in using basic calculus and more elegantly if you recognize spherical and

$Z = 2$, a better estimate for the binding energy of helium atom is 5.5 Ryd. [15 points]

The variational approach in QM is to take a plausible wave function and apply it to H' and tweak the wave function to yield the lowest value of E'. In this spirit, we recognize that each electron screens the nucleus for the other electron. Let Z_* be the effective charge seen by each electron. So, our tweaked wave function is a 1s wave-function appropriate for a 1s electron orbiting around a nucleus of charge Z_*, $\phi_{1\mathrm{s},Z*}(r)$. *The Hamiltonian still remains the one given by Equation 1.* Analytically compute the eigenvalue for this Hamiltonian with these new wave-functions. The answer is $(2Z_*^2 + {}^5\!/_4 Z^* - 4ZZ_*)$ Ryd. Minimizing this with respect to Z_* yields a binding energy of $(2Z^2 - {}^5\!/_4 Z + {}^{25}\!/_{128}$ Ryd of -77.45 eV. [15 points]

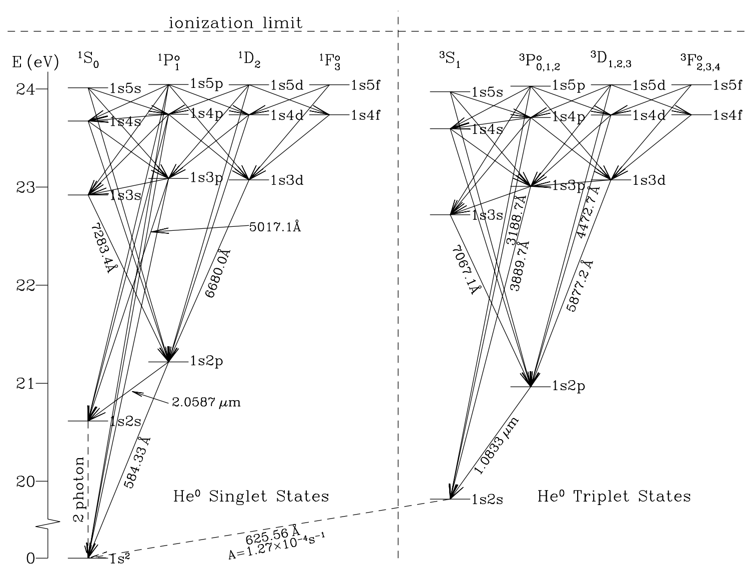

Figure 1: Grotrian diagram for He I (from Draine's ISM book).

[3] Inter-combination lines. Transitions between spin-families are called as "inter-combination" (aka "semi-forbidden") lines. For instance, transition between the single and triplet ladders of helium are inter-combination lines. Let us focus on the the first two excited states, 1s2s and 1s2p. The most famous "semi-forbidden" line is the decay from 1s2s $^3\mathrm{S}_1$ to the ground state (gs), 1s^2 $^1\mathrm{S}_0$ (A-coefficient is one of the smallest, $1.27\times10^{-4}\,\mathrm{s}^{-1}$. A less famous line is 1s2p $^3\mathrm{P}_{0,1,2} \to$ 1s2s $^1\mathrm{S}_0$. Consult NIST/ADS and list A-coefficients for these two lines and for transitions in the same spin ladder (that is, 1s2p $^3\mathrm{P}_{0,1,2} \to$ 1s2s $^3\mathrm{S}_1$ and 1s2p $^1\mathrm{P}_1 \to$ 1s2s $^1\mathrm{S}_1$). Explain clearly the trend you see. [10 points]

particle symmetry.

[4] PSR J0437−4715.[2] PSR J0437−4715 is the nearest (parallax of 6.43 ± 0.04 milliarcsecond) millisecond (spin period of 5.75 ms; known to 15 digits) pulsar. The measured dispersion measure is $2.64 \pm 3 \times 10^{-5}\,\mathrm{cm^{-3}}$ pc and the measured rotation measure is $0.155 \pm 0.018\,\mathrm{rad\,m^2}\,\mu$G. Compute the mean density and mean magnetic field. What are the limitations to these estimates? [5 points]

[5] The nearest major star-forming region. H II regions are ionized nebulae and particularly prominent when powered by young O stars in star-forming regions. Within the H II region the ionization fraction is essentially close to unity. Their primary cooling is via metastable lines (fine structure) of metals. The temperature of H II regions varies from 6,000 K (metal rich) to 9,000 K (metal poor). The nearest major H II regions to us is the Orion nebula. θ^1 Ori C is one of the bright stars in Orion, is embedded in an H II region with diameter of 0.5 pc and a mean electron density of $3200\,\mathrm{cm^{-3}}$. Assume a mean temperature of 8,000 K and mean distance to Orion of 400 pc. Compute the emission measure ($\mathrm{cm^{-6}}$ pc) for a line-of-sight through the center of the spherical H II region.

Compute the flux density (Jansky) from this HII region, resulting from free-free emission, at frequencies of interest to Caltech radio astronomers: 50 MHz (LWA), 1400 MHz (DSA-110), OVRO 40-m (12 GHz) and 30 GHz (COMAP) and soon the Leighton Chajnantor Telescope (LCT; 300 GHz). [Hint: You can accurately estimate the flux in the limit of low and high optical depth. For intermediate case you can take the ready-n-rough approach of approximating the HII region by a cylinder or (BONUS) do it correctly using spherical calculus.] [20 points]

[6] The Galactic Warm Ionized Medium. The Galactic WIM was inferred from low-frequency absorption seen towards the diffuse synchrotron emission from the inner Galaxy (Ellis & Hoyle 1962). Wisconsin Hα Mapper (WHAM) observed recombination line (Hα) from this medium. Looking towards the Galactic pole, the inferred emission measure is approximately, $2\,\mathrm{cm^{-6}}$ pc. The vertical scale height of this gas is 1 kpc. Assume a mean temperature of 8,000 K. The WIM, via thermal bremsstrahlung, provides a foreground[3] Compute the brightness temperature towards the Galactic poles at the frequencies listed in the previous problem. [10 points]

[2] `https://www.atnf.csiro.au/research/pulsar/psrcat/`

[3] which annoys CMB astronomers attempt in their quest to study the cosmic background.