Ay 121: Homework 6

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[1]. Spin-orbit coupling (including relativistic correction). Recall that spin-orbit increases rapidly with Z. Here is the abundance of metals: O (537 ppm), C (295 ppm), Ne (93 ppm), N (74 ppm), Mg (44 ppm), Si (36 ppm), and Fe (35 ppm). Within this group, iron is special because it has the largest Z. Thus, iron provides the best diagnostics of the hottest plasma in the Universe. From NIST/ASD, determine the energy level of 2s and 2p of hydrogen-like iron (Fe^{+25}). The transition from 2p to 1s is the iron Kα. (a) Compare the energy difference between the two levels of 2p, due to spin-orbit coupling and relativistic effect, with the analytical estimate (Dirac formula). (b) The Japan-US X-ray Imaging & Spectroscopy Mission (XRISM) carries a micro-calorimeter offering a non-dispersive energy resolution of 5 eV over the range 1.7–12 keV. Can XRISM resolve this doublet? [10 points]

[2] Relative intensities for fine structure transitions.[1] The spectrum of an alkali has three fine-structure components that correspond to the transitions ${}^2\mathrm{P}_{3/2} \rightarrow {}^2\mathrm{D}_{3/2}$, ${}^2\mathrm{P}_{3/2} \rightarrow {}^2\mathrm{D}_{5/2}$, and ${}^2\mathrm{P}_{1/2} \rightarrow {}^2\mathrm{D}_{3/2}$. In the optically thin regime the photon intensities of these components are observed to be in a ratio of 1:9:5 in the optically thin regime. Show that these observations satisfy the rule that the sum of the intensities of the transitions to or from a given level is proportional to its statistical weight $(2J + 1)$. [10 points]

[3] Bright He I 1.0833 μm emission from the terrestrial thermosphere. Metastable helium is helium in the 1s2s ${}^3\mathrm{S}_1$ level. There is only one path to decay, that is, to the ground state, 1s^2 ${}^1\mathrm{S}_0$. The A-coefficient is very small, $1.27 \times 10^{-4}\,\mathrm{s}^{-1}$ (see Figure 1).

Consider a metastable helium atom in the upper thermosphere of Earth, height 650 km (SPHEREx orbit) and illuminated by the sun. Solar photons can excite metastable helium to, for instance, 1s2p ${}^3\mathrm{P}_{0,1,2}$ which decays rapidly by emitting a photon, albeit in random

[1] This is a very useful result and is worth remembering. See Appendix §A for background.

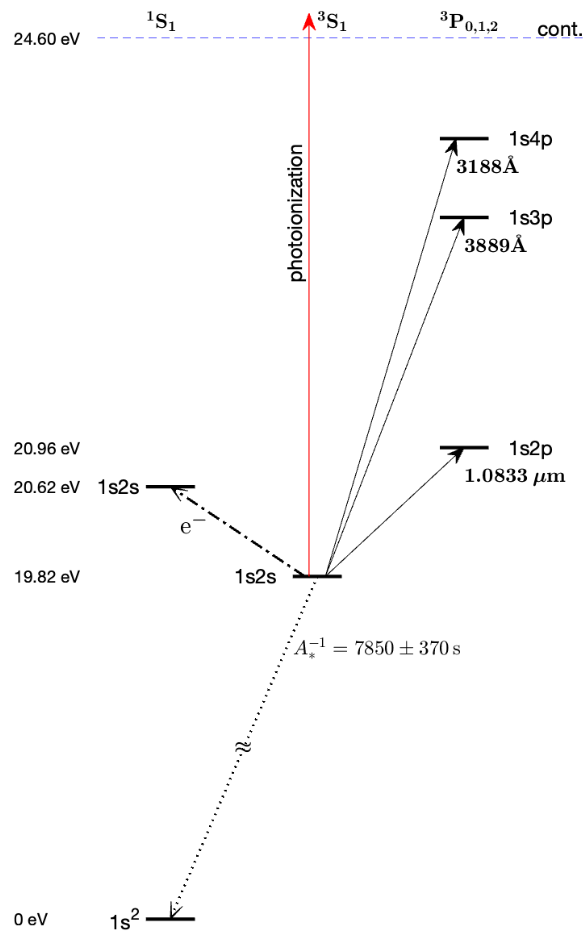

Figure 1 Partial Grotrian diagram of He I.

direction ("resonance scattering"). Occasionally, it could also be photo-ionized. Compute the scattering rate (number of scatterings per unit of time, per metastable atom) for the $1.083\,\mu$m transition. The necessary data can be found in §B. [20 points]

[4] Duration of the synchrotron flash. In §6.2 of Rybicki & Lightman (RL) the authors derive Δt^A, the duration of the synchrotron flash. In my view, Figure 6.2 of RL is misleading and the approximation presented ("$|\Delta \boldsymbol{v}| \approx v\Delta\theta$") just above Equation 6.8a is also misleading. The former is misleading because the trajectory of a relativistic particle follows a helix and, as such, is not confined to a plane (as implied by Figure 6.2). The latter is incorrect because the proper equations are given in Equation 6.3 and the approximation made by RL does not immediately follow from these two equations.

The simplest way to derive Δt^A is to recognize that the trajectory of the electron is a helix. In the sketch below, the observer (marked by a circled cross; top panel) is located in a plane perpendicular to the page and at a great distance away from the electron. Let us focus on point "L" in the figure below. Our goal is to calculate the distance between 1 and 2 as the electron goes through point L. The bottom panel shows the view from the top (and is meant to be the same as in RL).

Figure 2 The helical trajectory of a relativistic electron in a magnetic field (along the vertical axis).

Calculate κ, the radius of curvature of the helix.[2] Then the distance between 1 and 2 is simply $\kappa\Delta\theta$ where $\Delta\theta = 2/\gamma$. Show that this value agrees with that given by Equation 6.9 of RL. [15 points]

[5] Lifetime of a highly relativistic electron. Consider a highly relativistic electron with Lorentz factor $\gamma \gg 1$. It is moving in the interstellar medium with parameters typical of the case at the solar circle: the diffuse stellar photon field (characterized by a dilute blackbody with $T \approx 6,000\,\mathrm{K}$) with energy density $0.54\,\mathrm{eV\,cm^{-3}}$, the CMB ($0.26\,\mathrm{eV\,cm^{-3}}$) and magnetic field energy density of $0.9\,\mathrm{eV\,cm^{-3}}$.

Compute the time in which $\gamma = [10^4, 10^5, 10^6]$ electron will lose half of its energy due to inverse Compton scattering (CMB, starlight) and due to synchrotron radiation. [Warning: Do not use the Thompson cross section indiscriminately. Use an approximate cross section (Klein-Nishina) as needed.] 3×10 points

Compute the characteristic frequencies for each process. 10 points

[2] You may find §C to be very useful.

A The famous sodium yellow doublet

The ground state of Na I is $1s^2 2s^2 2p^6 3s$. In the early years of the *nineteenth century*, Joseph von Fraunhofer identified strong absorption lines in the spectrum of the sun and labeled them. Among them are D1 (5896 Å) and D2 (5890), which were later attributed to Na I. This pair arises from absorption of the ground term 3s $^2S_{1/2}$ to the first excited state: D1 ($^2S_{1/2} \rightarrow\ ^3P_{1/2}$) and D2 ($^2S_{1/2} \rightarrow\ ^3P_{3/2}$). Such allowed transitions from the ground state are called resonance lines. This line forms the basis for the sodium lamps that are used for street lighting.

If it were not for fine structure, the decay of all levels of 3p to 2s would not depend on m_l and m_s (which merely specify orientation). The L-S coupling leads to new wave functions that are based on linear combinations of (n, l, m_l, m_s) wave functions. Thus, the A-coefficient for D1 and D2 is the same (as is indeed the case; see NIST/ADS). One consequence is that if the excitation rate to the two levels of the 3p state is the same (cf. note the small energy difference between $^3P_{1/2}$ and $^3P_{1/2}$) then the emission from each levels is $\propto 2J + 1$ (at least in an optically thin situation). In case of the sodium D-doublet, the expected ratio 1:2 (D1:D2) is indeed seen (assuming no complications due to optical depth/radiative transfer).

B He I 1.0833 μm

The needed atomic physics is summarized in Table B and the solar spectral intensity can be found in Figure 3. For background information, see Kulkarni 2025 (`https://ui.adsabs.harvard.edu/abs/2025arXiv250914499K/abstract`).

C Helix

A helix can be specified by $x = a\cos(q)$, $y = a\sin(q)$, and $z = b$ where b is the distance gained along the z axis for every turn made. The unit vector of the tangent is given by $\boldsymbol{t} = d\boldsymbol{r}/ds$ where ds is the arc length; here, $\boldsymbol{r} = (x, y, z)$. For a parametric curve, $\boldsymbol{t} = d\boldsymbol{r}/dq/|d\boldsymbol{r}/dq|$. The curvature vector is $\boldsymbol{k} = d\boldsymbol{t}/ds$ which, for a parametric curve, is given by $d\boldsymbol{t}/dq/|d\boldsymbol{t}/dq|$. The radius of curvature is given by $\kappa = |\boldsymbol{k}|^{-1}$.

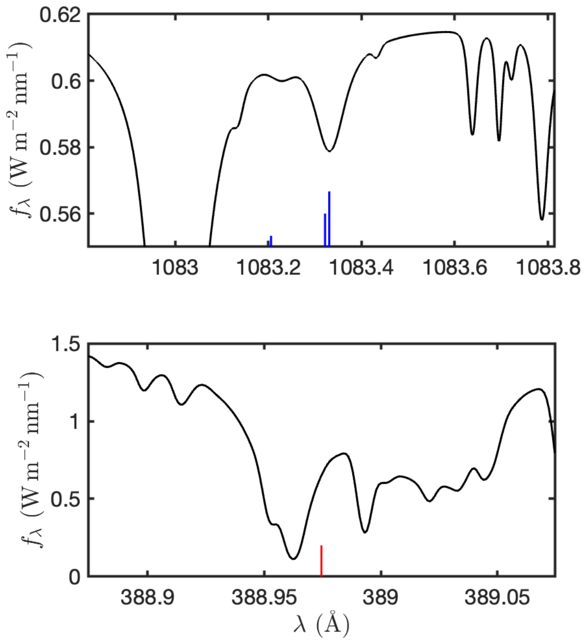

Figure 3 Zoom-in of the TSIS-1 HSRS solar spectrum centered on wavelengths of He I 1.0833 μm and He I λ3889. For the 1.0833 μm line, vertical blue stubs mark the three lines with height proportional to strength. For λ3889 the red vertical stub marks the effective wavelength of the triplet (see notes to Table B).

Table 1. Two major triplets of metastable He

λ (nm)	u	A_{ul} (s^{-1})	f_{lu}
1,083.205 7482	1s2p $^3\mathrm{P}^\mathrm{o}_0$	1.02×10^7	0.0599
1,083.321 6761	$^3\mathrm{P}^\mathrm{o}_1$	"	0.1796
1,083.330 6454	$^3\mathrm{P}^\mathrm{o}_2$	"	0.2994
388.970 65567	1s3p $^3\mathrm{P}^\mathrm{o}_0$	9.47×10^6	0.0072
388.974 75124	$^3\mathrm{P}^\mathrm{o}_1$	"	0.0215
388.975 08392	$^3\mathrm{P}^\mathrm{o}_2$	"	0.0358

Note. — λ is the wavelength of the transition. u is the upper state. For the lines listed here, the lower state (l) is 1s2s $^3\mathrm{S}_1$. A_{ul} is the A-coefficient of the ul transition and f_{lu} is the corresponding oscillator strength.