Clarification: Intensity and Pressure

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1 Motivating the concept of intensity

The concept of intensity is motivated by the following simple example. We start with Figure 1.6 of Rybicki & Lightman. Consider a star of radius R, luminosity L (total energy radiated per unit time). Let the observer be located at a distance r.

The *bolometric* (i.e., integrated over frequency) flux density at r is

$$F(r) = \frac{L}{4\pi r^2} \tag{1}$$

clearly is a function of r. In spherical coordinates, the differential solid angle $d\Omega = d\phi \sin(\theta) d\theta$. It is reasonable to define the solid angle of the star as

$$\Omega_s(r) = \int d\Omega = 2\pi \int_0^{\theta_c} d\theta \sin(\theta) = 2\pi[1 - \cos(\theta_c)] \tag{2}$$

where $\theta_c = \sin^{-1}(R/r)$ is the angle between the observer-star axis and the tangent line to the star's limb. Note that this definition of Ω_s is purely geometrical (i.e., the definition pays no attention to any brightness variation of the surface, in particular limb darkening).

For $\theta_c \ll 1$, $\cos(\theta) \approx 1 - \theta^2/2$ and $\sin(\theta) \approx \theta$ and, so $\theta_c \approx R/r$. In this limit,

$$\Omega_s(r) \approx \pi\theta_c^2 \ . \tag{3}$$

We see that Ω_s also falls off as r^{-2}. In contrast. the ratio

$$\mathcal{I} \equiv \frac{F(r)}{\Omega_s(r)} = \frac{1}{\pi} F(R) \tag{4}$$

is independent of r. Here, $F(R) = L/(4\pi R^2)$ is the flux density at the surface of the star.

This simple exam motivates intensity. It is an invariant as opposed to $F(r)$. In the textbook (Rybicki & Lightman, page 8) the authors provide a proof without resorting to the small angle approximation used above (but by demanding that the surface brightness of the star lack any angular dependence).

2 Radiation pressure

2.1 Pressure on the walls of the cavity

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We compute the pressure first at the wall of the cavity hosting the black body. Pressure is the rate of momentum deposited per unit time per unit area. From Figure we see that the horizontal momentum is canceled upon reflection whilst the perpendicular moment is doubled upon reflection. Thus,

$$P = \frac{dE}{dAdt} \quad = \quad \frac{1}{c}\int d\Omega I \cos(\theta)^2 \ . \tag{5}$$

For an isotropic radiation field,

$$P = \frac{2}{c}\int d\phi \int d\theta \sin(\theta)\cos(\theta)^2 = \frac{2\pi I}{3} \tag{6}$$

where the integration is from $\phi = [0, 2\pi]$ and $\theta = [0, \pi/2]$. The pressure can be compared with the energy density

$$U = \frac{1}{c}\int d\Omega I = \frac{4\pi I}{c} \tag{7}$$

where the integration is from $\phi = [0, 2\pi]$ and $\theta = [0, \pi]$. Thus $P = U/3$.

2.2 Pressure inside the cavity.

It is of some pedagogical value to compute the pressure at an arbitrary point within the cavity. Consider a small surface, dA. The momentum flux from the "top" cancels the momentum flux from the "bottom" and so one is tempted to conclude that the pressure is zero. Note that pressure is not the same as force. In fluids, force is due to pressure gradients. Naturally, the force anywhere inside a cavity with isotropic radiation field is zero.

Consider a bubble whose surface is entirely reflective. The force on dA of this sphere is $2\pi I/(3c)dA$. The flux of momentum coming into the bubble is $(A/c)\int d\Omega I \cos(\theta)^2$ where the integration is $\phi = [0, 2\pi]$ and $\theta = [0, \pi]$ and A is the surface area of the bubble. Pressure is the he momentum flux per unit area. The pressure on the bubble is thus the same as that given by Equation 6.