Choice of gauges: Coulomb & Lorenz

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During the Fall quarter of 2024 I had the occasion of teaching Radiative Processes (of interest to astronomers, aka Ay121) in the astronomy program at Caltech. This was the first time I have taught the course and, as usual, I was looking forward to the hit of adrenalin that comes from learning something new and understanding it and simultaneously the stress of explaining it to eager students. There is nothing more l like than learning[1] and testing myself by explaining it to someone else (step 1: offer to teach a new course; step 2: inquire things I do not understand with my dear friend Sterl Phinney and learn a lot from the feedback; step 3: attract an aspiring student, propose a a zany research program and finally succeed at it).

This M.O. has its genesis to a day at Caltech, long time ago. Sometime in the early eighties[2] I attended a class on General Relativity (GR) taught by Professor Richard Feynman at Caltech. The class was in one of the dank and dark rooms in Downs. I had read somewhere that Feynman had remarked "If you want to research, then teach; if you want to teach, then research". I guessed that Feynman was teaching this basic graduate class because he was probably looking into unifying GR with Quantum Mechanics (QM). I was really struck by Feynman's enthusiasm and strategy for new ventures.

Now let me jump to four decades, the present time. Every year the Astronomy Executive Officer[3] has to assign faculty to teach classes. I make the EO's job easier by allowing the EO to treat my course allocation as a free parameter. Many times this leads me to volunteering to teach new classes especially with demanding physics (at my level; this

[1]My motto: learn one new idea every day

[2]During the period 1978-1983, as a graduate student at University of California, Berkeley (UCB) I would occasionally visit my chum, Milind Purohit (now Dean, OIST, Okinawa, Japan) who was a graduate student in the Experimental Particle Physics group at Caltech. Milind used to live in one of the (then) graduate houses at the corner of Holliston and San Pasqual.

[3]Caltech has six Divisions such as Physics, Mathematics and Astronomy (the oldest). Each Division has "options" such as Physics, Mathematics and Astronomy and the leader of the option is the Executive Officer (EO) for that option. It is my observation that the EO is a geometric mean between the janitor of the building and the Chairperson of a traditional department. I can say this with considerable confidence since I served as the EO for Astronomy, 1997–2000.

immediately excludes quantum field theory and the like). I spend about 10 hours per lecture (first time around) but, like early morning jogs and cold showers, the experience has made a better person. I regret not initiating this scheme in my thirties.

With this introduction completed I now come to the main topic of this musing: a deficiency of graduate textbooks in astronomy. Most astrophysical graduate level books are written by theorists who, naturally, are deeply familiar with a wide range of theoretical framework. This familiarity means that many statements are made in the book that are obvious to the author but not to rusty tyros like me. Some authors even skip rigor with "trust me bro" statements ("it can be shown..." or worse, still, "it is trivial to show ..."). As such it is frustrating for an earnest observer like me to be satisfied by typical graduate level astronomy textbooks.

Below[4] I provide one such example. The point I am making is that authors, when writing a book, should get in the mind of the student and aim to excite, intrigue and finally give clear insight to the the student. Most readers of graduate astronomy books are observers whereas, as noted above, most writers are theorists. **I urge that all astro-theorists get an earnest observer to read the draft of their textbook and point areas where an additional exposition would be helpful to observers** who, after all, discover the truth about the Universe!

1 Gauge Choices

As with all first time teaching this has been a challenging and deeply satisfy experience, thus so far. [The dreaded four vectors and tensor algebra, emitted and received power, are up in a few weeks.]

The basic forces in electromagnetism are the electric field ($\mathbf{E}$) and the magnetic field ($\mathbf{B}$). Two well known results of vector algebra are: $\nabla \times \nabla\phi = 0$ for any scalar function, ϕ and $\nabla \cdot \nabla \times \mathbf{A} = 0$ for any vector function, $\mathbf{A}$. In electrostatics, $\nabla \times \mathbf{E} = 0$. Thus, we can assign a scalar potential, ϕ to the electric field, $\mathbf{E} = -\nabla\phi$. In magneto-statics as well as in Maxwell's model, $\nabla \cdot \mathbf{B} = 0$. So we can assign a vector potential to the magnetic field, $\mathbf{B} = \nabla \times \mathbf{A}$. However, the potentials are not uniquely defined. For example $\phi' = \phi + b$ where b is a constant is an equally valid potential as is $\mathbf{A}' = \mathbf{A} + \nabla\psi$. Both ϕ' and $\mathbf{A}'$ give rise to the same electric and magnetic fields.

In Maxwell's electromagnetic theory, substituting $\nabla \times \mathbf{A}$ for $\mathbf{B}$ is Faraday's equation results in

$$\nabla \times \big(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\big) = 0\ .$$

[4] Incidentally figuring out the material discussed below took me a good several hours.

Thus, in Maxwell's model, the electric field now has an associated scalar and vector potential:

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi \ .$$

However, as before, we have some freedom in defining the potentials. Specifically,

$$\mathbf{A}' = \mathbf{A} + \nabla\psi \ \ \& \ \ \phi' = \phi - \frac{1}{c}\frac{\partial\psi}{\partial t}$$

will leave the electric and magnetic fields unaffected.

Many text books (e.g., Rybicki & Lightman) simply state "The two common choices are Coulomb gauge, $\nabla\cdot\mathbf{A}' = 0$ and Lorenz gauge, $\nabla\cdot\mathbf{A}' + (1/c)\partial\psi/\partial t = 0$". For a person like me, an observer who has worked hard to reach this point, these statements are certainly intriguing and, simultaneously mysterious and overall, very unsatisfied.

After a fair bit of digging around and some contemplation I understood the rationale which, below, I first explained to myself and later explained to my Ay121 students. I wonder why textbooks do not add another paragraph along the lines below.

Coulomb gauge. The Coulomb gauge $\nabla\cdot\mathbf{A}' = 0$ demands

$$\nabla^2\psi = -\nabla\cdot\mathbf{A}.$$

This is Poisson's equation with a source term (RHS). *Solutions to this equation always exist* (Green's functions). So, we can legitimately make choice of Coulomb gauge.

Lorenz[5] gauge. The choice, $\nabla\cdot\mathbf{A}' + (1/c)\partial\psi/\partial t = 0$ leads to

$$\nabla^2\psi - \frac{1}{c}\frac{\partial^2\psi}{\partial t^2} = -\nabla\cdot\mathbf{A} - \frac{1}{c}\frac{\partial\phi}{\partial t}$$

The LHS is the space-time Laplacian operating on ψ and is a space time wave equation. The RHS is a source term. We have a wave equation with a forcing function. *Solutions always exist* (Green's function). Thus, we can always find ψ that justify the Lorenz choice.

I very much hope that logic along these lines can be incorporated in the next version of Rybicki & Lightman. Finally, by the time I finished this write up I was wondering what other gauge choices can be made!

[5]Not to be confused, as is frequently done, with Hendrik Lorentz of the famous Lorentz transforms, the Lorentz force, the Lorentzian, the Lorentz-Lorenz relation and Zeeman splitting fame.

A Postscript

One of the nice things at Caltech, despite my winsome personality, is that I have many friends across the spectrum of research methodologies. I circulated this musings to a select group and as can be seen from the response I have already benefitted from doing so.

Elias Most informed (paraphrasing): *A generalized Coulomb gauge,* $\nabla\cdot\mathbf{A} = \Gamma(\mathbf{A}, ..., \text{but not } \nabla\mathbf{A})$ *when applied to GR enabled numerical relativity (one of the key developments for the success of LIGO).*

Next, Most also pointed that the Lorenz gauge results in a wave equation for $\nabla\cdot\mathbf{A}$*:*

$$\Big[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\Big]\big(\nabla\cdot\mathbf{A}\big) = 0\ .$$

Sterl Phinney added: *Yes, the need to ensure a gauge that is guaranteed to have a solution, as by Green's function, is a critical step, which guarantees that Coulomb and Lorenz gauges can always be realized.*

If you have't previously, you might enjoy also understanding why it is so critically convenient to solve wave equations in an odd number of space dimensions (e.g. 1, 3; you will have notices that elementary texts always study waves and Green's functions in 1 and 3 dimensions, and never in 2 dimensions). Huygen's principle and the Lienard-Wiechart potential fail miserably in 2-D.

p.s. The fact that 2-d waves have dispersion even in vacuum or when like sound waves, their 3-d counterparts are dispersion less ("anomalous dispersion") is part of the reason for the long rumble of thunder, and the annoying loudness of shocks from supersonic airplanes. It is also important to the weirdness of 2-d materials like graphene. Courant & Hilbert Methods of Mathematical Physics Vol 2 is a good reference!