Ay 121: Final Exam

Instructor: S. R. Kulkarni

December 13, 2024

Framework. The mid-term is an oral exam lasting up to 60 minutes. The exams will be conducted over three days, December 11, 12 and 13, 2024. The master list for the oral exam is given below.

Logistics. *A Google sign-up sheet will be provided on the last day of the class. Sign up one of the slots. Please come to my office ONE hour before your appointment time. Pick up one of the paper slips (each of which has five questions). This is a closed-book exam. After collecting the slip you wend off to your office and work on the questions (no access to books, notes or equivalent information). You are advised to write down your answers on paper. At the appointed hour you come to my office and explain your answers to me (using the black board). Experience has taught me that a student who has* **written neatly (and also step-by-step analysis)** *does well when explaining their answer on the black board. Experience has taught me that it is wise for you to* **print this master list of questions**.

Changes to the master list since I first put it up on the web portal are marked by violet color.

Please note that an oral exam is very different from a written exam. The discussion stops when (1) we run out of time or (2) when examiner runs out knowledge! The grading is more indicative, in my view, of the student's true understanding. Also, an oral exam is also an opportunity for teaching and discussions usually lead to improved understanding.

You are allowed to use calculators. Ideally, you should know a number of basic constants but for this exam I am providing the values for constants and even basic formulae.

For reasons of fairness please do not discuss the exam with other students until the last student has finished their exam. You can discuss the experience in general terms, though.

Notes: Questions with a red title are longer than the others. You will be offered a total of five questions of which two or three will be long.

1. Intensity. Consider a star with radius R of uniform surface intensity (B) with radius R. Show that the flux at any distance, $r \geq R$, is

$$F = \pi B \left(\frac{R}{r}\right)^2$$

.

Background: §1.3 of RL.

2. Geo-coronal Hα. The Earth's thermosphere (located at a heigh of about 500 to 1,000 km) is heated by solar photo-dissociation of O_2 to temperatures between 1,000 °C to 2,000 °C. H (from H_2O and CH_4) and He escape ballistically, forming the exo-sphere of Earth.[1] The exosphere fluoresces, lit by solar EUV, and one of the consequence is "geo-coronal" Hα emission. At midnight, geocoronal Hα varies between 2 Rayleigh to 4 Rayleigh (depending on solar activity). Compute the intensity of Hα in photon/cm^2/s/arcsecond^2. What is the width of this line in Å? How does this compare with the dark sky from, say, Mauna Kea.

3. Ionizing rate. Consider a source which is putting Q ionizing photons per second. Let σ be the cross-section (cm^2) for a hydrogen atom to absorb a photon and get ionized. Write down the expression for the time scale for ionization at distance, d.

4. Einstein A and B coefficients.[2] Derive the relation between Einstein A and B coefficients.

5. Wien's Formula. Show that if stimulated emission is neglected, leaving only two Einstein coefficients, an appropriate relation between the coefficients will be consistent with thermal equilibrium between the atom and a radiation field of a Wien spectrum, but not of a Planck spectrum.

6. Wien's Displacement Law. The blackbody intensity can be expressed as $B_\nu(T)$ [as preferred by IR astronomers] or as $B_\lambda(T)$ [as preferred by optical astronomers]. The peak of the black body intensity distribution, B_ν corresponds to $h\nu_p = 2.82 k_B T$ whereas the peak of B_λ corresponds to $\lambda_p T = 0.29$ cm. Explain why $\lambda_p \nu_p \neq c$. What is a better way to estimate the frequency of peak radiation?

[1] The study of exo-spheres of extra-solar planets, particularly hot Jupiters, a major focus area of exo-planetology.

[2] You need to be able to construct Planck's formula from basic principles. Rote memorization will not be accepted.

7. Dimming of galaxies. Galaxies are extended objects and their apparent angular size depends on the sensitivity of the instrument to surface brightness. Using, a well known invariance, show that the total (bolometric) intensity of cosmological objects scales as $(1+z)^{-4}$.

8. Eddington Limit. Consider a main sequence star with luminosity, L ($\mathrm{erg\,s^{-1}}$) of radius R and mass, M. Assume that the photosphere is dominated by hydrogen, most of which is ionized. Consider an electron in photosphere of the star. Due to Thompson scattering the electron will experience a force pushing it away from the star. Gravity attracts it. At what luminosity will the radiative force be balanced by the gravitational force?

9. Recasting Maxwell's Equations. Maxwell's equations are centered on electric and magnetic field. These fields can be expressed as space and time derivatives of a vector potential ($\boldsymbol{A}$) and a scalar potential (ϕ). Show that the four equations of Maxwell can be recast to

$$\boldsymbol{\nabla}^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{1}{c}\frac{\partial}{\partial t}\Big(\boldsymbol{\nabla}\cdot\boldsymbol{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\Big) = -4\pi\rho \quad (1)$$

$$\boldsymbol{\nabla}^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \boldsymbol{\nabla}\Big(\boldsymbol{\nabla}\cdot\boldsymbol{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\Big) = -4\pi\boldsymbol{j} \quad (2)$$

Background: §2.5 of RL.

10. Gauge Choice. Continuing from the previous problem: the derivatives of potential are observables (field). So there is freedom to some degree in defining potentials ("gauge" choice). A popular choice is to set $\nabla\cdot\boldsymbol{A}+(1/c)\partial\phi/\partial t = 0$, which then simplifes Equations 1 and 2 to

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho, \qquad \nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -4\pi\boldsymbol{j} \quad (3)$$

and whose solution, in turn, lead to the Lienard-Wiechart potentials. Justify that this is a mathematically allowed choice. *Background: §2.5 of RL and, in particular, see SRK notes of October 15.*

11. Stoke's Parameters. Write down the normalized Stokes parameters ($[I, Q, U, V]/I$ for a linearly polarized signal (e.g. a pulsar), a circularly polarized signal (e.g., Jupiter-Io system, stellar radio flares from late type stars, e.g., GJ 1151) and an incandescent lamp.

12. Polarization due to Thompson scattering. Consider an electron located at the origin of a Cartesian frame. A circularly polarized electromagnetic wave propagating in the $\hat{z}$ axis is incident on the electron. The observer is located in y-z plane (unit vector: $\boldsymbol{n}$). Let $\cos(\theta) = \hat{z}\cdot\boldsymbol{n}$. You should be able to describe the parameters of the scattered wave as a function of θ.

13. Radiation Field. Consider the simple case of a charged particle that is accelerating along the the z axis ($\dot{u}$) with negligible velocity, $u/c \ll 1$ (Figure 1). Be prepared to estimate the strength and the direction of the electric and magnetic field at point P (far away from the origin) in the figure below. [See Appendix for the formula for electric and magnetic fields.]

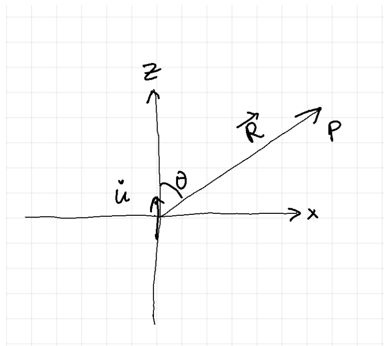

Figure 1: The geometry of an accelerated particle that is radiating.

14. Relativistic Larmor's formula. The non-relativistic Larmor's formula is given by

$$P = \frac{2q^2}{3c^3}\boldsymbol{a}\cdot\boldsymbol{a}$$

where $\boldsymbol{a}$ is the ordinary (3-vector) acceleration of the particle with charge q. The goal here is to derive the relativistic Larmor's formula. Place yourself in the frame (K') in which an accelerated electron can be regarded to be at rest albeit for a brief period, $\delta t'$. Since it is still accelerated it emits dipole radiation of energy dW' and the power, $P' = dW'/dt'$ is given by the non-relativistic Larmor's formula (above equation).

$$P' = \frac{2q^2}{3c^3}\boldsymbol{a}'\cdot\boldsymbol{a}'$$

where $\boldsymbol{a}'$ is the 3-vector acceleration of the particle in frame K$'$. The net linear momentum of of the emitted photons is zero, given the symmetry of dipole emission. Transform dW' and dt' to the observer frame and show that $P \equiv dW/dt = P'$. Conclude by showing that the covariant form of Larmor's formula is given by

$$P = \frac{2q^2}{3c^3}\vec{a}\cdot\vec{a}$$

where $\vec{a}$ is the four acceleration. *Background: §4.7 and §4.8.*

15. Continuity Equation. Derive the conservation of mass for a fluid (density, ρ; velocity, $\boldsymbol{v}$) in both Eulerian and Lagrangian frameworks.

16. Group Velocity. The dispersion relation for waves in the ocean is given by

$$\omega^2 = gk \tanh(kh).$$

where g is the acceleration due to gravity and h is the depth of the ocean floor. Derive the group and phase velocity in the limiting case of shallow and deep ocean.

17. Dispersion Measure. Assuming the dispersion relation for cold plasma derive the formula which gives the delay of a signal at frequency ν relative to infinite frequency. We will now apply this formula to the case of a a powerful Galactic magnetar that, on 28 April 2020, emitted a millisecond burst in X-rays and also radio. The former was detected by several space craft while the latter by CHIME (via side-lobe) and by Caltech graduate student Chris Bochenek's STARE2 project at OVRO at 1.4 GHz. The radio signal arrived about 0.7 s later relative to the X-ray signal. What is the inferred dispersion measure? [units: $\mathrm{cm}^{-3}\,\mathrm{pc}$.]

18. Free-free emission. In the usual formulation of free-free emission, only the emission from electrons curving around ions is included. Why is the emission from an electron curving away from another electron not included? Under what conditions would electron-electron free-free emission become important?

19. Electron-positron pair plasma. Some astrophysical sources put out electron-positrons pairs. Assume a nebula composed only of electrons and positrons with temperature of, say, $T = 10^7\,\mathrm{K}$ (admittedly, this is a bit colder than realistic examples). Compared to a ionized hydrogen blob of the same density and particle density will you see more or less free-free emission (and by what factor?).

20. Creation of e^+-e^- pairs. A positron colliding with an electron results in production of two photons. Here, we consider the inverse reaction, $\gamma + \gamma \to \mathrm{e}^+ + \mathrm{e}^-$. Specifically, consider two photons, one with direction along the x-axis (energy, E_1) and the other at an angle θ with respect to the x-axis (energy, E_2). They collide and the result is an electron and positron. Derive the conditions (in terms of E_1 and E_2) for this reaction to proceed.

21. Basic Synchrotron Emission. Consider an electron with Lorentz factor $\gamma \gg 1$ gyrating in the diffuse ionized medium of our Galaxy ($B \approx 3\,\mu\mathrm{G}$). Set the pitch angle, $\alpha = \pi/2$. Derive the characteristic frequency of the resulting synchrotron emission, ν_c. Next, by regarding the magnetic field as composed of virtual photons with angular frequency, $\omega_B = eB/m_e c$ show that the total synchrotron power loss is $-dE/dt \approx \gamma^2 \sigma_T c U_B$ where $U_B = B^2/(8\pi)$; here $E = \gamma m_e c^2$. Compute the timescale, $t_{1/2}$, over which an electron loses half its energy. Finally, setting $\gamma = 10^7$ compute ν_c and $t_{1/2}$.

22. Compton Scattering. A photon with energy E is incident on an electron is at rest in the laboratory frame. The angle between the incident and scattered photon is θ. Derive a formula for the energy of the scattered electron (E_s) and E_e, the kinetic energy of the scattered electron.

23. Greisen-Zatespin-Kuzmin limit. The highest energy cosmic ray ("Oh-My-God" particle), detected in 1991, had an estimated energy of 3.2×10^{20} eV. It is believe that this particle originated from an extra-galactic source. For simplicity assume it a proton (for simplicity) compute the Lorentz factor, γ. As the proton traverses intergalactic space it sees the CMB. Compute the energy of a typical CMB photon as seen in head-on collision within the frame of the OMG particle. Unlike an electron, a proton is a complicated particle, and, as a result, there are several outcomes for the interaction of an energetic photon and a proton: $p^+ + \gamma \rightarrow p^+ + e^+ + e^-$ production (important at lower energies) and pion production $p^+ + \gamma \rightarrow p^+ + \pi^0$ or $p^+ + \gamma \rightarrow n + \pi^+$ (important at higher energies). Assuming that pion (π^+, π^0) production dominates (with cross-section given in Figure 2) compute the mean free path of the OMG particle due to scattering by CMB.

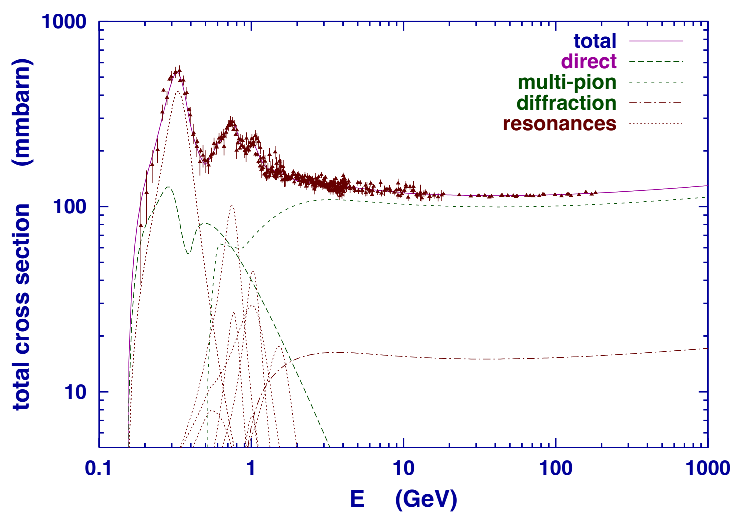

Figure 2: The proton-pion cross-section. Please note that "mmbarn" is μbarn!! `https://ui.adsabs.harvard.edu/abs/2006hep.ph....7265A/abstract`

24. FeXXVI Lyα. Compute the wavelength of the equivalent Lyα for Fe^{+25}. What is the speed and orbital radius of the electron in ground state?

25. Terms resulting from LS-coupling. Consider carbon. Work out, from basic principles, the terms of the ground electron configuration ($1s^2 2s^2 2p^2$). *Background: Review November 7.*

26. Hund's rules. Write down the various terms of the ground electron configuration of O^0 ($1s^2 2s^2 2p^4$). Apply Hund's rule to determine the ground level. *Background: Review*

SRK Notes of November 7.

27. Critical Density. Consider a 2-level atom in which only collisions and spontaneous emission matter (cf. see my notes of October 8). Derive the critical density in this framework. The ground electron state of C^+ has two levels (split by spin-orbit coupling; see Figure below). The transition between is the famous 157.7 μm fine structure line. From Draine's ISM book (§17.4) we find that the collisional de-excitation coefficient (from upper state "1" to lower state "0") due to electrons is $k_{10} = 4.5 \times 10^{-8} T_4^{-1/2}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$ Using your formulation compute the critical density of electrons.

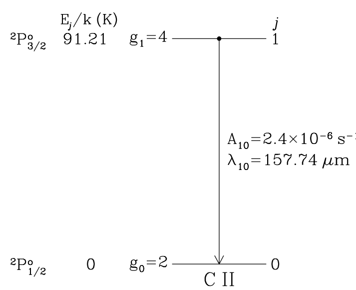

Figure 3: The ground term of C^+.

28. Selection Rules. List the selection rules for dipole transitions (in roughly decreasing order of importance). *Background: See SRK notes of December 3.*

29. Allowed, Semi-Forbidden & Forbidden Transitions. Please explain why these lines are classified as allowed, forbidden, semi-forbidden and forbidden transitions. Provide the rationale for the classification. Separately, comment on transition (E). Corrected typos related to electron configuration.

(A) N II 1084 Å $1s^2 2s^2 2p^2\ ^3P_0 - 1s^2 2s^2 2p3s\ ^3D^o_1$

(B) N II] 2143 Å $1s^2 2s^2 2p^2\ ^3P_2 - 1s^2 2s 2p^3\ ^5S^o_2$

(C) [N II] 6549 Å $1s^2 2s^2 2p^2\ ^3P_1 - 1s^2 2s^2 2p^2\ ^3D_2$

(D) [N II] 205 μm $1s^2 2s^2 2p^2\ ^3P_0 - 1s^2 2s^2 2p^2\ ^3P_1$

(E) $2s\ ^{1/2}S_{1/2} - 1s\ ^{1/2}S_{1/2}$

Background: §6.7 of Draine's book on ISM. For your convenience I provide a Grotrian diagram for N II (Figure 4).

30. Hyperfine lines. ^{4}He, ^{12}C and ^{16}O are amongst the most abundant elements, following hydrogen. Do you expect to see hyperfine lines from these elements?

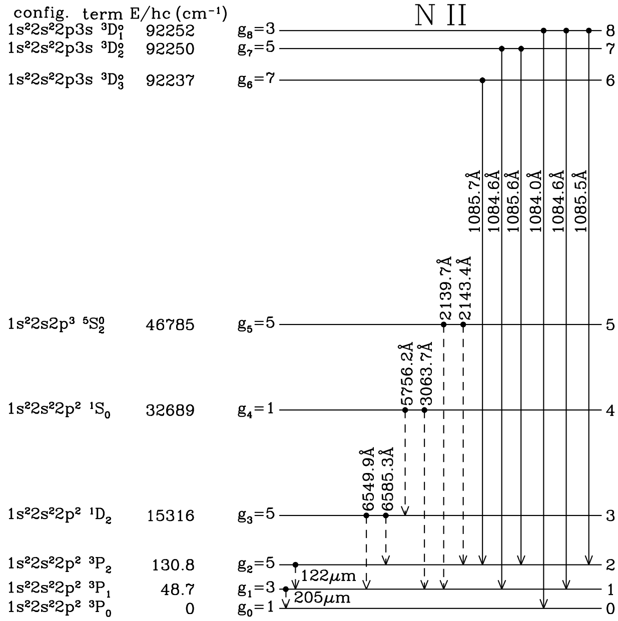

Figure 4: Grotrian diagram for N II (from Draine §6.7).

31. Diatomic molecules. Historically, the molecular medium was probed using the rotation lines of $^{12}C^{16}O$. Why not the far more abundant, H_2?

32. HD and H_2. Please compare the rotation, vibration and electronic transition of HD and H_2.

Appendix: Constants, Formulae and Data

All constants are CGS or Astro-CGS (e.g., $\mathrm{cm}^{-3}\,\mathrm{pc}$), unless otherwise noted.

$c = 2.998 \times 10^{10}\,\mathrm{cm\,s^{-1}}$
$h = 6.63 \times 10^{-27}\,\mathrm{CGS}$
$N_A = 6.02 \times 10^{23}$
$e = 4.8 \times 10^{-10}\,\mathrm{CGS}$ [$e = 1.6 \times 10^{-19}$ Coulomb, SI]
$m_p/m_e = 1836$
$G = 6.67 \times 10^{-8}\,\mathrm{GCS}$
statvolt(CGS) = 299.8 volt(SI)
1eV=1.6×10^{-12} erg, $k_B T = 0.86 T_4\,\mathrm{eV}$
Ry $= R_\infty = 2\pi^2 e^4 m_e/h^3 \approx 13.6\,\mathrm{eV} \rightarrow 109,737.316\ \mathrm{cm}^{-1}$ (Rydberg)
$\sigma_T = (8\pi/3) r_e^2 = 0.66\,\mathrm{barn}$ (1 barn=$10^{-24}\,\mathrm{cm}^{-2}$; $r_e = e^2/(m_e c^2)$ is the electron radius)
$a_0 = \hbar^2/(m_e c^2) = 0.53\,\text{Å}$ (Bohr radius, 1Å=10^{-8} cm)
Pion Mass: π^0 (135 MeV), $\pi^\pm$ (140 MeV)

radian, 206,265 arcseconds (2.1×10^5)
parsec, 3×10^{18} cm
Mass of Sun, $M_\odot = 2 \times 10^{33}$ gram
Radius of Sun, $R_\odot = 7 \times 10^{10}$ cm (or 2 light seconds)
Luminosity of Sun, $L_\odot = 4 \times 10^{33}\,\mathrm{erg\,s^{-1}}$
Astronomical unit is 500 light seconds

Sky brightness: On a clear dark night at Mauna Kea the sky brightness, in the optical band, is roughly 2 Rayleigh per Å.

Maxwell's equation: These equations should be known to any graduate student of astronomy. However, for many of you, CGS is new and so I am providing the Lorentz force and the equations of Maxwell along with some useful vector algebra relations:

$$
\begin{aligned}
\boldsymbol{F} &= q[\boldsymbol{E} + (\boldsymbol{v}/c) \times \boldsymbol{B}] \\
\nabla \cdot \boldsymbol{E} &= 4\pi\rho \\
\nabla \cdot \boldsymbol{B} &= 0 \\
\nabla \times \boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \\
\nabla \times \boldsymbol{B} &= \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \ .
\end{aligned}
$$

Useful vector relations:
$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$ ("bac minus cab")

$$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}, \qquad \nabla \times \nabla(\psi) = 0, \qquad \nabla \cdot (\nabla \times \boldsymbol{A}) = 0,$$

Radiation Field. From §3.2 we see that electric and magnetic field of an accelerated particle of charge q, at distance R, are given by

$$\boldsymbol{E}(\boldsymbol{r}, t) = q\left[\frac{\boldsymbol{n} - \boldsymbol{\beta}(1-\beta^2)}{\kappa^3 R^2}\right] + \frac{q}{c}\left[\frac{\boldsymbol{n}}{\kappa^3 R} \times \{(\boldsymbol{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\}\right] \tag{4}$$

$$\boldsymbol{B}(\boldsymbol{r}, t) = \boldsymbol{n} \times \boldsymbol{E}(\boldsymbol{r}, t) \tag{5}$$

where $\boldsymbol{\beta} = \boldsymbol{u}/c$, $\kappa = 1 - \boldsymbol{n} \cdot \boldsymbol{\beta}$ and $\boldsymbol{n} = \boldsymbol{R}/R$ (see Figure 1 above). The

"[...]"

refer to retarded time.

Blackbody Radiation:
Energy density, $u(T) = aT^4$, $a = 7.56 \times 10^{-15}\,\mathrm{erg\,cm^{-3}}$
Emergent flux, $F = \pi B(T) = \sigma_{SB} T^4$, $\sigma_{SB} = ac/4$.
The mean energy, $\langle E \rangle$ of a blackbody radiation field is $2.8\,k_B T$.

Plasma Frequency:

$$\omega_p^2 = \frac{4\pi n_e e^2}{m_e} \tag{6}$$