Material Derivative

The Eulerean view of fluid mechanics

$$V(x,y,z,t) \qquad \rho(x,y,z,t)$$

However forces act on particles (Lagrangian framework)

$$\vec{a}_{particle} = \frac{d}{dt}\left(V_{particle}(x,y,z,t)\right)$$

$$= \frac{\partial \vec{v}}{\partial t}\frac{dt}{dt} + \frac{\partial \vec{v}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{v}}{\partial y}\frac{d\vec{y}}{dt} + \frac{\partial \vec{v}}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{v}}{\partial x}v_x + \frac{\partial \vec{v}}{\partial y}v_y + \frac{\partial \vec{v}}{\partial z}v_z$$

$$= \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}$$

Represent

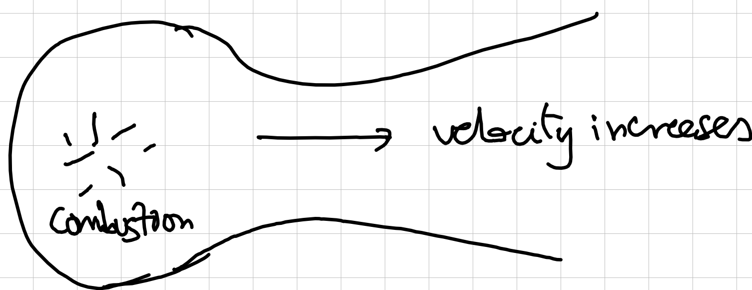
$$\frac{\partial}{\partial t} + \vec{v}\cdot\nabla \rightarrow \frac{D}{Dt}$$ (material derivative)

$$\vec{a} = \frac{D}{Dt}(\vec{v})$$

$\frac{D}{Dt}$ can be applied to other variables

Note in steady flow $\frac{\partial}{\partial t} = 0$ but you can still have acceleration

example:



Laval nozzle

Lecture 13: Propagation in plasma

$$\nabla \cdot E = 4\pi\rho \qquad \nabla \cdot B = 0$$

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t} \qquad \nabla \times B = \frac{4\pi j}{c} + \frac{1}{c}\frac{\partial E}{\partial t}$$

$$j = n_i q_i \vec{v}_i + n_e q_e v_e$$

Equations of fluid mechanics.

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \vec{v}_j) = 0 \qquad j = e, i$$

$$m_j n_j \frac{D\vec{v}_j}{Dt} = q_j n_j \left[E + \frac{1}{c} v_j \times (B + B_0)\right] - \nabla P_j$$

B_0 is the static magnetic field while B is the magnetic field of the EM wave.

P = pressure $= n_j k_B T$

(and in a plasma $T_i \neq T_e$, necessarily)

The energy conservation equation should be included but a common approach is to assume a "constitive relation" ex $P = f(\rho)$ "polytrope"

$$\nabla \times (\nabla \times E) = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times B)$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left(4\pi j + \frac{\partial}{\partial t}E\right)$$

but $\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E$

So,

$$\nabla(\nabla \cdot E) - \nabla^2 E = -\frac{1}{c}\frac{\partial}{\partial t}\left(4\pi j + \frac{\partial E}{\partial t}\right)$$

Consider a wave $E \propto e^{i\omega t - i\vec{k}\cdot\vec{x}}$

$$\dot{\vec{E}} = -i\omega\vec{E} \qquad \nabla^2\vec{E} = -k^2\vec{E}$$

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\vec{E} - \vec{k}(\vec{k}\cdot\vec{E}) = \frac{4\pi i\omega}{c^2} j$$

In vacuum there are no sources of charge

So RHS is zero.

$$\nabla \cdot E = 4\pi\rho = 0 \Rightarrow \vec{k}\cdot\vec{E} = 0$$

Thus $k^2 = \frac{\omega^2}{c^2}$ and $\vec{k}$ is perpendicular to $\vec{E}$

Next, $\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$

$\Rightarrow \quad \vec{k} \times \vec{E} \propto \vec{B}$

This $\vec{E}$ is perpendicular to $\vec{B}$ (even in the presence of charges)

- In vacuum, k, E, B are perpendicular to each other
- In plasma, E is perpendicular to B but k is not necessarily $\perp$ to $\vec{E}$

- Let us simplify by ignoring the heavy ions
- We will then "linearize" the equations
  keep only $O(\frac{v}{c})$ but not $O(\frac{v^2}{c^2})$
- Assume pressure can be ignored ("cold" plasma)
- For now, $B_0 = 0$

$$m_e \frac{\partial \vec{v_e}}{\partial t} = -e\left(\bar{E} + \frac{\bar{v_e}}{c} \times \bar{B}\right)$$

If the electrons have small velocities then we can ignore $v \times B$

$$\therefore \quad m_e \frac{\partial \vec{v}}{\partial t} = -e\vec{E}$$

with $v \propto e^{i\omega t - ik.x}$

$$\Rightarrow \quad \vec{v_e} = \frac{e\vec{E}}{im_e\omega}$$

plugging into $\vec{j} = n_e \vec{v_e} e$

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\vec{E} - k(k.E) = \frac{4\pi i\omega}{c^2}\frac{n_e e^2 E}{im_e\omega}$$

$$= \frac{4\pi}{c^2}\frac{n_e e^2}{m_e}\vec{E}$$

$\nabla \cdot E = 4\pi\rho = 0$

$\Rightarrow k.E = 0$

$$k^2 - \frac{\omega^2}{c^2} = \frac{4\pi n_e e^2}{m_e c^2}$$

$$\omega^2 = k^2c^2 + \omega_p^2 \qquad \omega_p^2 = \frac{4\pi n e^2}{m_e}$$

Had we included the ions of charge Z

$$\omega^2 = k^2c^2 + \underbrace{\omega_p^2}_{\frac{4\pi}{m_i} n_i Z^2 e^2 + \frac{4\pi n_e e^2}{m_e}}$$

The phase velocity is defined to be

$$v_{ph} = \frac{\omega}{k} \equiv \frac{c}{n_R}$$

refractive index

So

$$n_R = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

Note for $\omega > \omega_p$, $n_R < 1$ and so $v_{ph} > c$

$\omega < \omega_p$, k becomes imaginary

so, wave dies down.

Group velocity:

Consider two phasors $k_0 \pm \Delta k$ $\quad v_p = \frac{\omega_0}{k_0}$

$\omega_0 \pm \Delta\omega$

The sum of the electric fields is

$$E_t = e^{i(k_0-\Delta k)x - i(\omega_0-\Delta\omega)t} + e^{i(k_0+\Delta k)x - i(\omega_0+\Delta\omega)t}$$

$$= 2e^{ik_0(x-v_p t)}\cos(\Delta k\, x - \Delta\omega t)$$

The velocity which keeps the amplitude constant is

$$x = \frac{\Delta w}{\Delta t} t$$

~~Let~~

$$\frac{\Delta w}{\Delta k} = \frac{\partial w}{\partial k} + \frac{\Delta k}{2}\frac{\partial^2 w}{\partial k^2} + \cdots \Big|_{k=k_0}$$

Keeping only the first term

$$v_g = \frac{\partial w}{\partial k}$$

For our case

$$v_g(\omega) = c\left(1 - \frac{\omega^2}{\omega_p^2}\right)$$

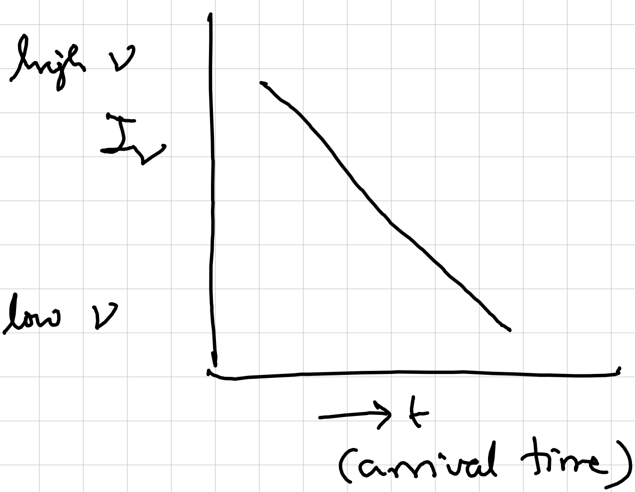
Arrival time

$$t(\nu) = \int_0^L \frac{1}{v_g(l)}\, dl$$

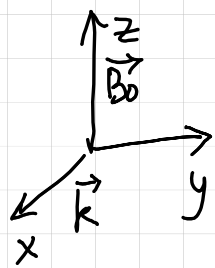
$$\simeq \int_0^L \frac{dl}{c}\left(1 + \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right)$$

$$t(\nu) - t(\infty) = \tau(\nu) = \frac{e^2 \times \text{parsec}}{2\pi m_e c}\frac{1}{\nu^2} DM$$

$$DM = \int \underbrace{n_e}_{/\mathrm{cm}^{-3}} \underbrace{dl}_{\mathrm{pc}}$$



Magnetized Plasma:



ordinary   extra-ordinary

"O"   "X"

↕   ↔

$$\boxed{\omega_c = \frac{eB_0}{mc}}$$

(a) $\vec{k} \perp \vec{B_0}$: (Ordinary)

Assume incident linear polarized signal ($\hat{z}$)

$$\vec{E} = E_0 \hat{z}$$

In this case, $v \times B_0 = 0$.

$\Rightarrow$ so, no change in dispersion relation

"ordinary"

(b) $\vec{k} \perp \vec{B_0}$: Extra-ordinary

Assume, now, $\vec{E} = E_0 \hat{y}$

In this case

$$\frac{c^2 k^2}{\omega^2} = \frac{c^2}{v_p^2} = 1 - \frac{\omega_e^2}{\omega^2}\left(\frac{\omega^2 - \omega_e^2}{\omega^2 - \omega_h^2}\right)$$

$$\omega_h^2 = \omega_e^2 + \omega_c^2 \qquad \text{"hybrid frequency"}$$

Cutoff frequencies: $k = 0$

Resonce frequencies: $k \to \infty$

Two cutoff frequencies:

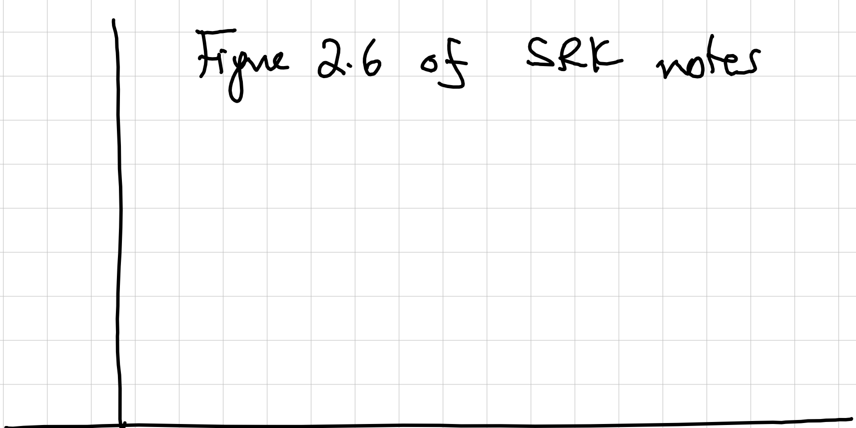
$$\omega_R = \frac{1}{2}\left[\left(\omega_c^2 + 4\omega_e^2\right)^{1/2} + \omega_c\right]$$

$$\omega_L = \frac{1}{2}\left[\left(\omega_c^2 + 4\omega_e^2\right)^{1/2} - \omega_c\right]$$

Resonace:

$$\omega = \omega_h$$

The extra-ordinay wave has two bands for transmission (SRK notes)



$\vec{k} \parallel \vec{B}_0$: Faraday rotation and whistler waves:

Here, we have to allow for E_x, E_y

$$E = E_0 \left[\cos(\omega t)\,\hat{x} \pm \sin(\omega t)\,\hat{y}\right]$$

(see SRK notes for RCP, LCP definition)

"A signal is called as Right-handed circular polarization (RCP) if, when viewed along the $\vec{k}$ axis, the electric vector traces a clockwise circle" [IEEE]

For LCP:

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_e^2}{\omega(\omega+\omega_c)}$$

This has a cutoff at $\omega = \omega_L$

Behaves like unmagnetized case but with cutoff of ω_L instead of ω_e

For RCP:
$$\frac{c^2k^2}{\omega^2} = n_R^2 = 1 - \frac{\omega_e^2}{\omega(\omega-\omega_c)}$$

This has a cutoff $\omega = \omega_R$

resonance $\omega = \omega_c$

Figure 2-7: Whistler waves.

Let $\vec{E}_{RCP} = E_0 \cos\omega t \, \hat{x} + E_0 \sin\omega t \, \hat{y}$

$\vec{E}_{LCP} = E_0 \cos\omega t \, \hat{x} - E_0 \sin\omega t \, \hat{y}$

We see that $\vec{E}_{RCP}$ rotates counter-clockwise

$\vec{E}_{LCP}$ rotates clockwise.

We define phase by the direction of the electric field at $t=0$. Specifically, if along $\hat{x}$ then phase $= 0$

At $t=0$, $\vec{E}_{RCP} = E_0 \hat{x}$

$\vec{E}_{LCP} = E_0 \hat{x}$

Thus, the phase of both RCP and LCP is 0

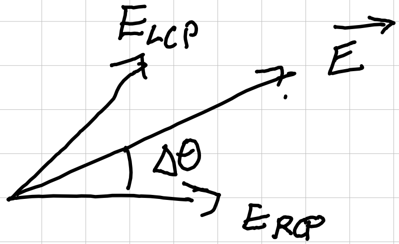
Now, consider the sum: $\vec{E} = \vec{E}_{RCP} + \vec{E}_{LCP}$

$= 2 \cos\omega t \, \hat{x}$

Thus, the sum of RCP and LCP, each with phase $= 0$ is a linearly polarized signal along $\hat{x}$ axis.

Now, consider the case when, say, LCP has advanced by phase, $\Delta\phi$, while RCP has not.

$t = 0$



$$\vec{E} = \vec{E}_{RCP} + \vec{E}_{LCP}$$

$$= E_0\,\hat{x} + E_0 \cos(\Delta\phi)\,\hat{x} + E_0 \sin(\Delta\phi)\,\hat{y}$$

$$= E_0 \left[(1+\cos\Delta\phi)\,\hat{x} + \sin\Delta\phi\,\hat{y} \right]$$

We see

$$\tan\Delta\theta = \frac{\sin\Delta\phi}{1+\cos\Delta\phi}$$

$$= \frac{2\sin\left(\frac{\Delta\phi}{2}\right)\cos\left(\frac{\Delta\phi}{2}\right)}{2\cos^2\left(\frac{\Delta\phi}{2}\right)}$$

$$\therefore \tan\Delta\theta = \tan\phi/2 \Rightarrow \Delta\theta = \phi/2$$

$\Rightarrow$ the linearly polarized signal is rotated by $\dfrac{\Delta\phi}{2}$

Faraday rotation:

We now focus on propagation $\omega \gg \omega_c$
ω_e

$$k = \frac{c}{\omega}\sqrt{n_R} = \frac{c}{\omega}\sqrt{1 - \frac{\omega_e^2}{\omega(\omega \pm \omega_c)}}$$

$$k_{R,L} \simeq \frac{\omega}{c}\left[1 - \frac{\omega_e^2}{2\omega^2}\left(1 \mp \frac{\omega}{\omega_e}\right)\right]$$

The difference in k results in phase-shift between RCP and LCP

$$\phi_{R,L} = \int_0^d k_{R,L}\, dl$$

$\Delta\theta$ = angle by which linearly polarized signal is rotated

$\frac{1}{2}$ (see above)

$$= \frac{1}{2}\int_0^d (k_R - k_L)\, ds$$

$$\Delta\theta = \frac{2\pi e^3}{m^2 c^2 \omega^2}\int_0^d n B_{\parallel}\, ds$$

$$\Delta\theta = 0.81 \cdot \left(\frac{\lambda}{1\,\mathrm{m}}\right)^2 \cdot \underbrace{\int n_e \left(\frac{B}{\mu G}\right) \frac{dl}{pc}}_{RM}$$