Lecture 7:

3.5 Radiation Reaction

$$P = \frac{2q^2\dot{u}^2}{3c^3} \qquad \text{Larmor's formula}$$

On a timescale $\quad T \simeq \frac{\frac{1}{2}mv^2}{P} = \frac{3}{2}\frac{mc^3}{e^2}\left(\frac{u}{\dot{u}}\right)^2$

Let $t_p \simeq \frac{u}{\dot{u}}$, the deceleration timescale

$$T = \frac{t_p}{\tau}\, t_p$$

$$\tau = \frac{2e^2}{3mc^2} \simeq \frac{r_0}{c} \sim 10^{-23}\,\mathrm{s} \qquad \left[\frac{e^2}{r_0} = mc^2\right]$$

If $\frac{T}{t_p} \gg 1$ then $\frac{t_p}{\tau} \gg 1$.

So, we can regard losses to be gradual when

$$\frac{t_p}{\tau} \gg 1.$$

On longer timescales, radiation reaction should matter.

A simple approach is

$$-\vec{F}_{rad} \cdot \vec{u} = \frac{2e^2 \dot{u}^2}{3c^3}$$

However, LHS depends on $\vec{v}$ whereas RHS has no $\vec{u}$ dependence.

This violates Galilean relativity.

$$-\int_{t_1}^{t_2} \vec{F}_{rad} \cdot \vec{u}\, dt = \frac{2e^2}{3c^3} \int_{t_1}^{t_2} \dot{\vec{u}} \cdot \dot{\vec{v}}\, dt$$

$$= \frac{2e^2}{3c^3} \left[\underbrace{\dot{u} \cdot u \Big|_{t_1}^{t_2}}_{\text{non-radiation field} \doteq 0} - \int_{t_1}^{t_2} \ddot{u} \cdot u\, dt \right]$$

$$\vec{F}_{rad} = \frac{2e^2 \ddot{v}}{3c^3} = m \tau \ddot{u} \qquad \tau = \frac{2e^2}{3c^3}$$

radiation reaction force

3.b Harmonically Bound particle

Consider a classical harmonic oscillator (but with charge). This will radiate

$$\vec{F} = -k\vec{r} = -m\omega_0^2 \vec{r}$$

$$m\ddot{\vec{r}} = -k\vec{r}$$

Include radiation reaction

$$-\tau \dddot{x} + \ddot{x} + \omega_0^2 x = 0$$

Assume that the motion is largely harmonic

$$\dddot{x} \simeq -\omega_0^2 \dot{x}$$

$$\Rightarrow \quad \ddot{x} + \omega_0^2 \tau \dot{x} + \omega_0^2 x = 0$$

[c.f. pendulum in air.

also pendulum in air can have legitimate friction as $\propto \dot{x}$ but not in relativity]

Solve by assuming $x \propto e^{\alpha t}$

$$\alpha^2 + \omega_0^2 \tau \alpha + \omega_0^2 = 0$$

$$\alpha = \pm i\omega_0 - \frac{1}{2}\omega_0^2 \tau + O(\omega_0^3 \tau^2)$$

Note $\omega_0\tau \ll 1$ $\qquad \frac{\omega_0}{2\pi} = \nu_0 \sim 10^{15}\,Hz$, say

$\tau \sim 10^{-23}\,s.$

So $\omega_0 \tau \sim 10^{-8}$

$$x(t) = x_0 e^{-\Gamma t/2} \cos(\omega_0 t)$$

$$= \frac{1}{2} x_0 \left[e^{-\frac{1}{2}\Gamma t + i\omega_0 t} + e^{-\frac{1}{2}\Gamma t - i\omega_0 t} \right]$$

$$\Gamma = \omega_0^2 \tau = \frac{2e^2\omega_0^2}{3mc^3}$$

The Fourier transform is

$$\hat{x}(\omega) = \frac{1}{2\pi} \int_0^\infty x(t) e^{i\omega t}\, dt$$

$$= \frac{x_0}{4\pi} \left[\frac{1}{\frac{\Gamma}{2} - i(\omega+\omega_0)} + \frac{1}{\frac{\Gamma}{2} - i(\omega-\omega_0)} \right]$$

negative $\qquad$ positive frequency

Owing to Hermetian symmetry it is sufficient to stick to positive frequency

$$|\hat{x}(\omega)|^2 = \left(\frac{x_0}{4\pi}\right)^2 \frac{1}{(\omega-\omega_0)^2+\left(\frac{\Gamma}{2}\right)^2}$$

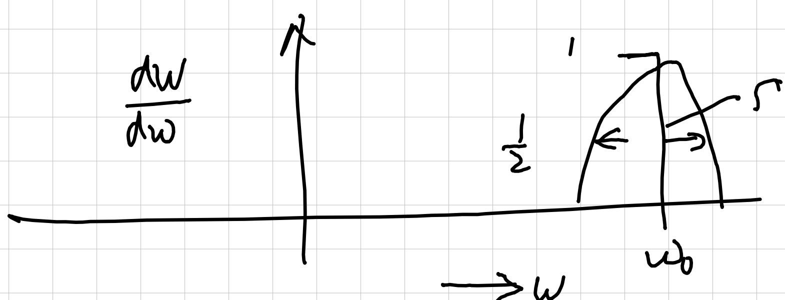
By the dipole formula the power spectrum is

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}|\hat{d}(\omega)|^2$$

dipole

$$\left[cf: \quad d(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\,\hat{d}(\omega)\,d\omega \right] \quad d(t) = -e\,x(t)$$

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}\,\frac{e^2 x_0^2}{(4\pi)^2}\,\frac{1}{(\omega-\omega_0)^2+\left(\frac{\Gamma}{2}\right)^2}$$



Can rewrite it as

$$\frac{dW}{d\omega} = \left(\frac{1}{2}k x_0^2\right)\left[\frac{\Gamma/2\pi}{(\omega-\omega_0)^2+(\Gamma/2)^2}\right]$$

Lorentz profile

$$W = \int \frac{dW}{d\omega} d\omega = \frac{1}{2} k x_0^2$$

Note that

$$\Gamma = \omega_0^2 \tau$$

or

$$\frac{\omega_0}{\Gamma} = \frac{1}{\omega_0 \tau} = 2\pi \frac{P_0}{\tau}$$

P_0 ← period

τ ← damping time

$$\frac{P_0}{\tau} \gg 1$$

Incidentally, $\Delta\omega = \Gamma$ = line width in ω, when expressed in wavelength is a <u>constant</u>!

$$\lambda = 2\pi \frac{c}{\omega}$$

$$|\Delta\lambda| = 2\pi c \frac{\Delta\omega}{\omega^2} = 2\pi c \frac{\omega_0^2 \tau}{\omega_0^2}$$

$$= 2\pi c \tau = 1.2 \times 10^{-4} \text{Å}$$

Driven Harmonically Bound Particle:

$$m\ddot{x} = -m\omega_0^2 x + m\tau\dddot{x} + eE_0 \cos\omega t$$

As before, $x = x_0 e^{i\omega t}$

Obtain the power-spectrum, as before,

$$P(\omega) = \frac{e^2 |x_0|^2 \omega^4}{3c^3} = \frac{e^4 E_0^2}{3m^2c^3} \frac{\omega^4}{(\omega^2-\omega_0^2)^2 + (\omega_0^3\tau)^2}$$

The time-averaged Poynting vector is

$$\langle S \rangle = \frac{c}{8\pi} E_0^2$$

$$\sigma(\omega) = \frac{P(\omega)}{\langle S \rangle} = \sigma_T \frac{\omega^4}{(\omega^2-\omega_0^2)^2 + (\omega_0^3\tau)^2}$$

$\omega \gg \omega_0$

$\sigma(\omega) \to \sigma_T$ free-electron

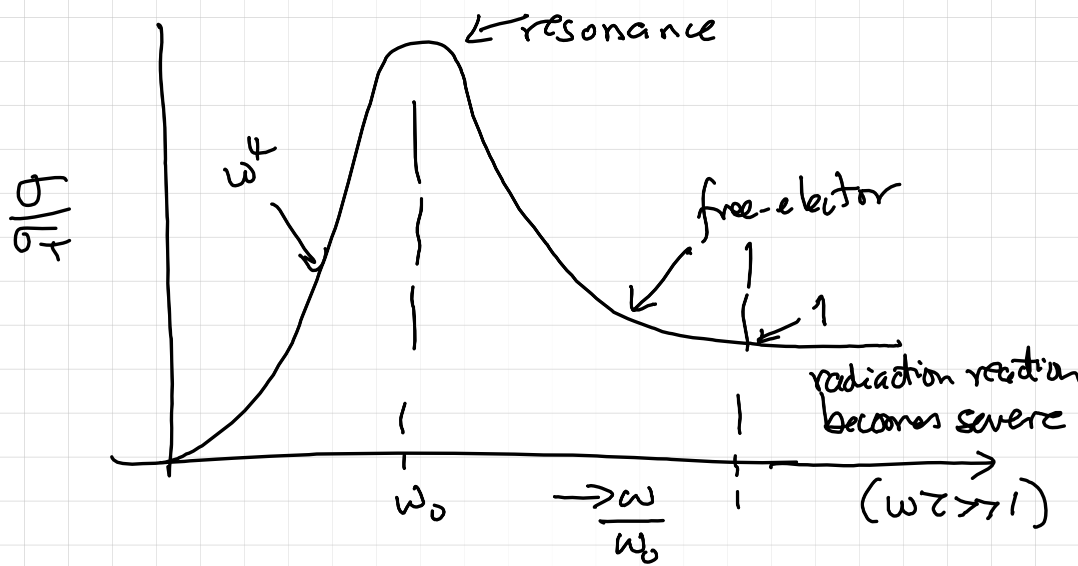
$\omega \ll \omega_0$

$\sigma(\omega) \to \sigma_T \left(\frac{\omega}{\omega_0}\right)^4$ Rayleigh scattering

In this limit, the electron has no inertia $m \to 0$

$$-m\omega_0^2 x + eE_0 \cos\omega t = 0$$

So the radiation results from the polarization of the medium by the incident electric field.



$\underline{\omega \simeq \omega_0}$

$$\omega^2 - \omega_0^2 \rightarrow (\omega - \omega_0)(\omega + \omega_0)$$

$$\sim (\omega - \omega_0) 2\omega_0$$

$$\sigma(\omega) \simeq \frac{\pi \sigma_T}{2\pi} \frac{\Gamma / 2\pi}{(\omega - \omega_0)^2 + (\Gamma/2)^2}$$

$$\Gamma = \omega_0^2 \tau$$

$$\sigma(\omega) = \frac{2\pi^2 e^2}{mc} \frac{\Gamma / 2\pi}{(\omega - \omega_0)^2 + (\Gamma/2)^2}$$

$$\Rightarrow \int \sigma(\omega)\, d\omega = \frac{2\pi^2 e^2}{mc}$$

$$\int \sigma(\nu)\, d\nu = \frac{\pi e^2}{mc}$$

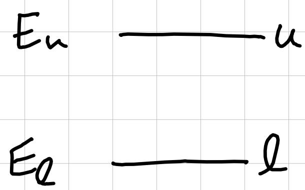
Quantum mechanics show that

$$\int \sigma(\nu)\, d\nu = f_{lu} \frac{\pi e^2}{mc}$$

oscillator strength

$f_{lu} \simeq 1$ for allowed transition

Lecture: 2-level atom with collisions



Absorption

$$\left(\frac{dn_u}{dt}\right)_{l\to u} = -n_l B_{lu} u_\nu \qquad h\nu = E_u - E_l$$

Emission:

$$\left(\frac{dn_l}{dt}\right)_{u\to l} = -n_u B_{ul} U_\nu - n_u A_{ul}$$

In steady state; these two should match

$$n_l B_{lu} u_\nu = n_u B_{ul} U_\nu + n_u A_{ul}$$

Applying this framework to black-body radiation and assuming statistical thermal equilibrium holds:

$$g_u B_{ul} = g_l B_{lu}$$

(using U_ν instead of $\bar{J}$)

$$B_{ul} = \frac{c^3}{8\pi h \nu^3} A_{ul}$$

Define: $\bar{n}_\gamma \equiv \frac{c^2}{2h\nu^3} \bar{I}_\nu = \frac{c^3}{2h\nu^3} u_\nu$

angle averaged

(photon occuption number)

$$\left(\frac{dn_l}{dt}\right)_{u\to l} = n_u \cdot A_{ul} (1 + \bar{n}_\gamma)$$

$$\left(\frac{dn_u}{dt}\right)_{l\to u} = n_l \frac{g_u}{g_l} A_{ul} \bar{n}_\gamma$$

Importance of stimulated emission relative to spontaneous emission.

Absorption Cross-section:

$cu = F$

$$-\left(\frac{dn_l}{dt}\right) = n_l B_{lu} u_\nu$$

Photon flux (all directions) $F = \frac{cu}{h\nu_{lu}}$

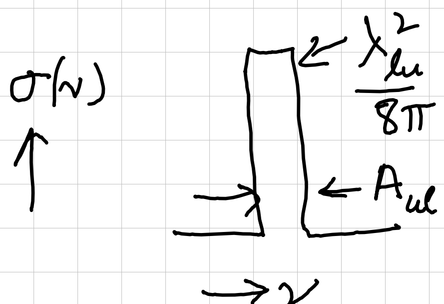
$$= n_l \frac{\sigma_\nu c u_\nu}{h\nu_{lu}}$$

$$B_{lu} = \frac{c}{h\nu_{lu}} \int \sigma_\nu \, d\nu$$

or equivalently

$$\int d\nu \, \sigma_\nu = \left(\frac{g_u}{g_l}\right) \frac{c^2}{8\pi \nu_{lu}^2} A_{ul}$$

$$= \left(\frac{g_u}{g_l}\right) \underbrace{\frac{\lambda_{lu}^2}{8\pi}}_{cm^2} \underbrace{A_{ul}}_{Hz}$$



So, the strength of the line is directly related to

(1) λ_{lu}^2

(2) A_{ul}, the bandwidth of the line

Oscillator Strength:

dimension-less

$$\underbrace{f_{lu}}_{\text{dimension-less}} \equiv \underbrace{\frac{m_e c}{\pi e^2}}_{cm^{-2} s^{+1}} \overbrace{\int \sigma_\nu \, d\nu}^{cm^2 s^{-1}}$$

$$\therefore \quad A_{ul} = \frac{8\pi^2 e^2 \nu^2}{m_e c^3} \frac{g_l}{g_u} f_{lu}$$

$$\boxed{A_{ul} = \frac{0.6670 \ cm^2 s^{-1}}{\lambda_{lu}^2} \frac{g_l}{g_u} f_{lu}}$$

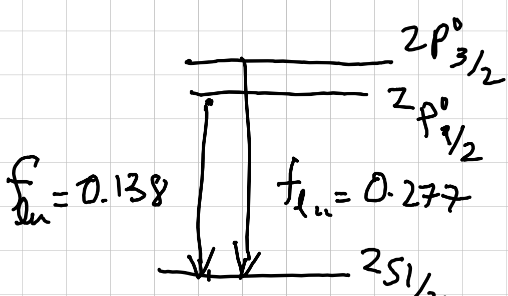
$f_{lu} \simeq 1$ "permitted" lines

$f_{lu} \ll 1$ "forbidden" line

ex. Lyα

A: $2.6 \times 10^8 \, s^{-1}$
(both)

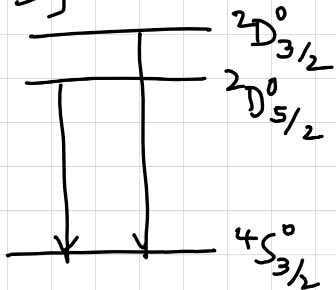
f_{lu}: 0.416
(quoted)



[OII]

A: 2.6×10^{-5}

f: 8.9×10^{-14}
$_{lu}$



A: 1.59×10^{-4}

f: 3.3×10^{-13}

With particle collisions:

k_{lu} = collisional excitation coefficient
$cm^3 s^{-1}$

k_{ul} = " de-excitation "

n_c = number density of colliders

Thermal equilibrium

$$n_l k_{lu} = n_u k_{ul}$$

but

$$\frac{n_u/g_u}{n_l/g_l} = e^{-E_{ul}/kT}$$

$$\therefore \boxed{g_l k_{lu} = g_u k_{ul} e^{-E_{ul}/k_B T}}$$

Assume no ambient radiation field.

$$n_l n_c k_{lu} = n_u n_c k_{ul} + n_u A_{21}$$

$$\frac{n_u}{n_l} = \frac{g_u}{g_l} \frac{e^{-E_{ul}/kT}}{1+\frac{n_{crit}}{n_c}}$$

n_{crit} (critical density): $\dfrac{A_{ul}}{k_{ul}}$

When

$n_c \gg n_{crit}$ $\qquad \dfrac{n_u}{n_l} = \dfrac{g_u}{g_l} e^{-E_{ul}/k_B T}$ ("thermal")

$n_c \ll n_{crit}$ $\qquad \dfrac{n_u}{n_l} = \dfrac{g_u}{g_l} \underbrace{\left(\dfrac{n_c k_{ul}}{A_{ul}}\right)}_{\ll 1} e^{-E_{ul}/k_B T}$

("sub-thermal")

$$\frac{n_u}{n_l} = \frac{k_{ul}}{A_{ul}} n_c$$

Photon emission rate per atom is

$$l = \frac{n_u A_{21}}{(n_u + n_l)} \simeq \frac{n_u}{n_l} A_{21}$$

$$\simeq k_{ul} n_c$$

Each collision leads to emission

At high density

$$l \approx \underbrace{\frac{g_u}{g_l} e^{-E_{ul}/kT}}_{\text{no dependence on } n_c} A_{ul}$$