Tensors:

Zeroth-rank tensor is a scalar
first-rank tensor is a 4-vector
second-rank tensor has 16 numbers

$T^{\mu\nu}$ $\quad \mu = 0, 1, 2, 3$
$\quad \nu = 0, 1, 2, 3$

$$\boxed{T_{\mu\nu} = \eta_{\mu\sigma} \eta_{\nu\tau} T^{\sigma\tau}}$$

Asscociated covariant

Mnemonic: "Co- is low and that is all you need to know"

Mixed component tensors

$$T^{\mu}{}_{\nu} = \eta_{\nu\tau} T^{\mu\tau} \qquad T_{\mu}{}^{\nu} = \eta_{\mu\sigma} T^{\sigma\nu}$$

Lorentz transformations:

$$T'^{\mu\nu} = \Lambda^{\mu}{}_{\sigma} \Lambda^{\nu}{}_{\tau} T^{\sigma\tau}$$

$$T'^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\sigma} \tilde{\Lambda}_{\nu}{}^{\tau} T^{\sigma}{}_{\tau}$$

$$T'_{\mu}{}^{\nu} = \tilde{\Lambda}_{\mu}{}^{\sigma} \Lambda^{\nu}{}_{\tau} T_{\sigma}{}^{\tau}$$

"The position of the tensor index, as a superscript or subscript, determines whether it is contravariant or covariant in its transformations"

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$

This is a second-rank tensor

$$F'_{\mu\nu} = \tilde{\Lambda}_\mu{}^\alpha \tilde{\Lambda}_\nu{}^\beta F_{\alpha\beta} \quad \text{(Lorentz boost)}$$

where

$$\tilde{\Lambda}_\mu{}^\nu = \eta_{\mu\tau} \Lambda^\tau{}_\sigma \eta^{\sigma\nu}$$

$\uparrow$ Lorentz boost

$$\Lambda^\mu{}_\nu = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Some properties:

- $F_{\mu\nu} F^{\mu\nu} = 2(B^2 - E^2) =$ invariant
- $\det(F) = (E \cdot B)^2 =$ invariant

Electromagnetic Force equation(s)

$$F^\mu = m_0 a^\mu = \frac{dp^\mu}{d\tau}$$

where $\quad a^\mu = \dfrac{dU^\mu}{d\tau} \quad$ (four-acceleration)

Lorentz force (spatial)

$$\vec{F} = e\left(\vec{E} + \frac{1}{c}\,\vec{v}\times\vec{B}\right)$$

This suggests that

$$\boxed{F^\mu = \frac{e}{c}\,F^\mu{}_\nu\,U^\nu}$$

Substituting for $F^\mu{}_\nu$ we obtain

$$\frac{dW}{dt} = e\,\vec{E}\cdot\vec{v} \qquad \text{(time)}$$

$$\frac{d\vec{p}}{dt} = e\left(\vec{E} + \frac{1}{c}\,\vec{v}\times\vec{B}\right) \qquad \text{(space)}$$

Relation between $\vec{U}$ and $\vec{a}$

$$\vec{U}\cdot\vec{a} = U_\mu \frac{dU^\mu}{d\tau}$$

$$= \frac{1}{2}\frac{d}{d\tau}\left(U_\mu U^\mu\right) = \frac{1}{2}\frac{d}{d\tau}\left(-c^2\right) = 0$$

Thus, the four-velocity and four-acceleration are orthogonal.

An equally general result is

"The four-force and four velocity are orthogonal, regardless of the origin of the force"

Electric & magnetic field transformation:

(A) $\quad \bar{E}'_{\parallel} = \bar{E}_{\parallel} \qquad \bar{E}'_{\perp} = \gamma(\bar{E}_{\perp} + \beta \times \vec{B})$

(B $\quad \bar{B}'_{\parallel} = \bar{B}_{\parallel} \qquad \bar{B}'_{\perp} = \gamma(\bar{B}_{\perp} - \beta \times \vec{E})$

- $B^2 - E^2$ is a Lorentz invariant

  $E \cdot B$ is a Lorentz invariant

Physical intution for equations (A), (B)

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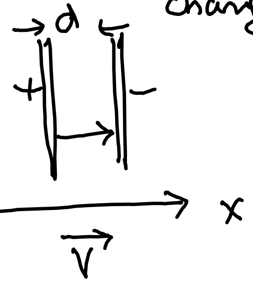
changed capacitor

$E_x = 4\pi\sigma$

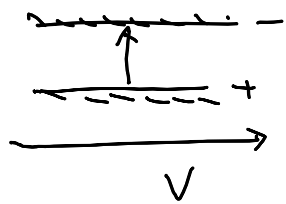


In K', the surface density σ is unchanged

the separation is d/γ

$E'_x = E_x$

Now rotate the capacitator



$E_x' = \gamma E_x$

$\Rightarrow \quad E_\perp' = \gamma E_\perp$

(1) σ is increased by γ.

(2) there is now a surface current $\sigma \gamma v$

$$\int B \cdot ds = \frac{4\pi}{c} \times \text{current enclosed by path}$$

$$B_z' = -\frac{4\pi}{c}(\sigma\gamma)v$$

$$\Rightarrow \quad B_\perp' = -\gamma \vec{\beta} \times E_\perp$$

Chapter 4.6

Fields of a moving charged particle

Consider a charged particle at rest in K'

Frame K' moves w.r.t to K by velocity $\vec{v}$

$$E'_x = \frac{qx'}{r'^3} \qquad B'_x = 0$$

$$E'_y = \frac{qy'}{r'^3} \qquad B'_y = 0$$

$$E'_z = \frac{qz'}{r'^3} \qquad B'_z = 0$$

$$r' = (x'^2 + y'^2 + z'^2)^{1/2}$$

We apply E-B transformations $K' \to K$

$$E_x = \frac{qx'}{r'^3} \qquad B_x = 0$$

$$E_y = \frac{q\gamma y'}{r'^3} \qquad B_y = -\frac{q\beta\gamma z'}{r'^3}$$

$$E_z = \frac{q\gamma z'}{r'^3} \qquad B_z = \frac{q\beta\gamma y'}{r'^3}$$

$$r = \sqrt{\gamma(x-vt)^2 + y^2 + z^2}$$

$$E_x = \frac{q\gamma(x-vt)}{r^3} \qquad B_x = 0$$

$$E_y = \frac{q\gamma y}{r^3} \qquad B_y = \frac{-q\gamma\beta z}{r^3}$$

$$E_z = \frac{q\gamma z}{r^3} \qquad B_z = \frac{q\gamma\beta y}{r^3}$$

Note that since $E' \cdot B' = 0$ in K' we have $\bar{E} \cdot \bar{B} = 0$ in frame K.

Thus $\vec{E}$, $\vec{B}$ are perpendicular.

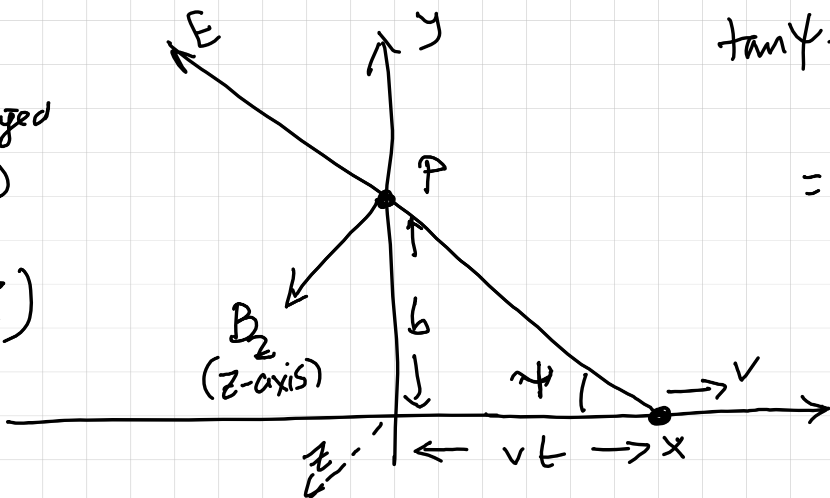
R&L show that this is consistent with Lienard-Wiechert result.

$$\vec{E} = \frac{q(\hat{n} - \vec{\beta})(1-\beta^2)}{K^3 R^2}$$

I could not arrive at this result!

(points to the charged particle)

$(\vec{B} \perp \vec{E})$



$\tan\psi = \dfrac{E_y}{E_x} = -\dfrac{b}{vt}$

Choose an observation point P $x=0, y=b, z=0$.

$$E_x = \frac{-q v \gamma t}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} \qquad B_x = 0$$

$$E_y = \frac{q \gamma b}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} \qquad B_y = 0 \qquad z=0$$

$z=0 \quad E_y = 0 \qquad B_z = \beta E_y$

Notice: (1) $\dfrac{E_y}{E_x} = -\dfrac{b}{vt}$

$\Rightarrow$ $\vec{E}$ points to source

(2) $\vec{B} \perp \vec{E}$

(3) in the limit of $\gamma \gg 1$, $|B| \simeq |E|$

The Power spectrum (EM waves)

§ 2.3 (pp. 58)

$$\hat{E}(\omega) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} E(t)\, e^{i\omega t}\, dt$$

with

$$E(t) = \int_{-\infty}^{\infty} \hat{E}(\omega)\, e^{i\omega t}\, dt$$

However, since $E(t)$ is real, $\hat{E}(\omega)$ enjoys Hermetian symmetry

The Poynting vector is the energy flux

$$\frac{c}{4\pi} E \times B = \frac{c}{4\pi} |E|^2$$ where $E \perp B$, $|E| = |B|$

So $$\frac{dW}{dA\,dt} = \frac{c}{4\pi} E^2(t)$$

(energy/area/time)

The total energy in the flux is

$$\frac{dW}{dA} = \frac{c}{4\pi} \int E^2(t)\, dt$$

Parseval's Theorem:

$$\int E^2(t)\, dt = 2\pi \int |\hat{E}(\omega)|^2\, d\omega$$

Using Hermetian symmetry this becomes

$$\int_{-\infty}^{\infty} E^2(t)\,dt = 4\pi \int_0^{\infty} |\hat{E}(\omega)|^2\,d\omega$$

So we have

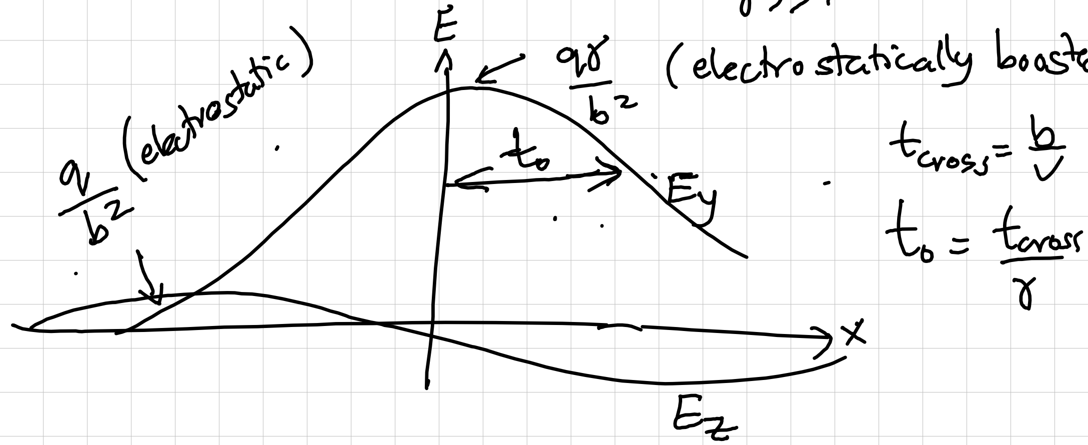
$$\frac{dW}{dA} = c\int_0^{\infty} |\hat{E}(\omega)|^2\,d\omega$$

This lead us to identify

$$\frac{dW}{dA\,d\omega} = c\,|\hat{E}(\omega)|^2$$

total energy per unit frequency ("power-spectrum") per unit area

For $\gamma >> 1$



Thus a relativistically moving charged particle leaves a perpendicular pulse boosted by γ $\left(E = \frac{q}{b^2}\gamma\right)$ and short pulse $\left(\Delta t = \frac{t_{cross}}{\gamma} \text{ where } t_{cross} = \frac{b}{v}\right)$

Say $b = 1\,\mathring{A}$ $\gamma = 10$, $(v \simeq c)$

$E \cdot l$ = work done = $\frac{\gamma q^2}{b^2} \cdot b = \frac{\gamma q^2}{b} = \gamma I_H \left(\frac{b}{1\mathring{A}}\right)^{-1}$

(ionization)

$\frac{1}{\Delta t} = \frac{\gamma}{b/v} = \frac{\gamma c}{b}$

$\lambda = \frac{c}{(1/\Delta t)} = c\Delta t = \frac{cb}{\gamma c} = b/\gamma$