A Crash course in statistical optics

ref: STATISTICAL OPTICS by J.W. Goodman (Wiley-Interscience)

Semiclassical Model for Photoelectric Detection

Three assumptions

① Probability of emission of a photoelection by a detector whose size (ΔA) is small compared to coherence area of the incident light, in time interval $(\Delta \tau)$ small compared to the coherence time ($\Delta \tau \gg 1/\nu_0$, however) is proportional to

$$\int I(x,y,t)\, dA\, dt$$

i.e. $$p(\Delta A, \Delta t) = \alpha \int I(x,y,t)\, dA\, dt$$

related to quantum efficiency

② For a sufficiently small, $\Delta A, \Delta t$, the probability of emission of more than one photon is negligible.

③ Photoelection process has no memory.
i.e. all the Δt event outcomes are independent

Under these assumptions

$$p(K) = \frac{\overline{K}^{K} e^{-\overline{K}}}{K!} \quad \text{where } \overline{K} = \alpha \overline{W}$$

$$\overline{W} = \langle \int dt \int dA \; I(x,y,t) \rangle$$

W is the integrated intensity.

MANDEL'S FORMULA

Above we assumed W is known and fixed. Mandel proposed

$p(K/W)$ = conditional prob. of K electron emission given W

$$= \frac{(\alpha W)^{K} e^{-\alpha W}}{K!}$$

If W is determined by a statistical process,

$$p(K) = \int_0^{\infty} p(K/W)\, p_W(W)\, dW$$

where $p_W(W)$ is the ~~conditional~~ probability distribution of W.

Some general results.

We know that for a Poisson process

$$E\big(K(K-1)\ldots(K-n+1)\big) = \overline{K}^{\,n}$$

Now consider Mandel's formula

$$p(K) = \int p(K|W)\, p_W(W)\, dW$$

The expectation of $K(K-1)\ldots(K-n+1)$ for $p(K)$ is

$$E\big(K(K-1)\ldots(K-n+1)\big) = \sum_{K=0}^{\infty} K(K-1)\ldots(K-n+1)\, p(K)$$

$$= \sum_{K=0}^{\infty} K(K-1)\ldots(K-n+1) \int_0^{\infty} \frac{(\alpha W)^K e^{-\alpha W}}{K!} p_W(W)$$

Interchange the summation and integration

$$= \int_0^{\infty} dW\, p_W(W) \sum_{K=0}^{\infty} \big(K(K-1)\ldots(K-n+1)\big) \frac{(\alpha W)^K e^{-\alpha W}}{K!}$$

$$= \int_0^{\infty} dW\, p_W(W)\, (\alpha W)^n$$

$$= \langle (\alpha W)^n \rangle$$

In particular

$$\bar{K} = \alpha \bar{W}$$

$$\overline{K(K-1)} = \overline{(\alpha W)^2}$$

or
$$\overline{K^2} - \bar{K} = \overline{(\alpha W)^2}$$

or
$$\overline{K^2} - \bar{K}^2 = \overline{(\alpha W)^2} - (\alpha \bar{W})^2 + \bar{K}$$

$$\boxed{\sigma_K^2 = \bar{K} + \alpha^2 \sigma_W^2}$$

Poisson "Detection" noise

Statistical noise due to variations in W ex Photocount statistics of well stabilized single mode laser radiation

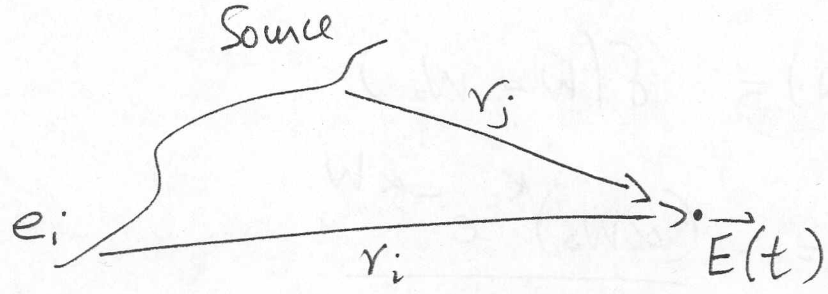
$$p_W(W) = \delta(W - W_0)$$

$$p(K) = \frac{(\alpha W_0)^K e^{-\alpha W}}{K!}$$

$$\boxed{\sigma_K^2 = \bar{K}}$$

$$SNR = \frac{\bar{K}}{\sigma_K} = \sqrt{\bar{K}}$$

ex Thermal radiation with $\Delta t << \tau_c$



$$\vec{E}(t) = \sum_{k=0}^{\infty} \frac{\vec{e}_k(r_k, t)}{r_k} e^{j 2\pi k r_k}$$ Huygens principle

$\vec{e}_i$ is the electric field ~~at~~ from radiator "i"

We argue $Re[E]$ and $Im[E]$ are Gaussian random variates, $\mu = 0$ and identical σ^2.

We can show (do this yourself)

$$cov\left[Re[E], Im[E] \right] = 0$$

$\Rightarrow$ $Re[E]$, $Im[E]$ are independent variates.

Thus $I(t) = E E^*$ ~~fo~~ has an exponential distribution

$$p_W(w) = \frac{1}{\bar{W}} e^{-W/\bar{W}} \qquad W \geq 0$$

$$\therefore \; p(K) = \int_0^\infty \frac{(\alpha W)^K}{K!} e^{-\alpha W} \frac{1}{\bar{W}} e^{-W/\bar{W}} \, dW$$

$$p(K) = \frac{1}{1+\alpha\bar{W}} \left(\frac{\alpha\bar{W}}{1+\alpha\bar{W}}\right)^K$$

with

$$\bar{K} = \alpha\bar{W}$$

$$p(K) = \frac{1}{1+\bar{K}} \left(\frac{\bar{K}}{1+\bar{K}}\right)^K$$

Bose Einstein Distribution

Geometric distrib.

One can show

$$E\big(K(K-1)\cdots(K-n+1)\big) = n!\,\bar{K}^n$$

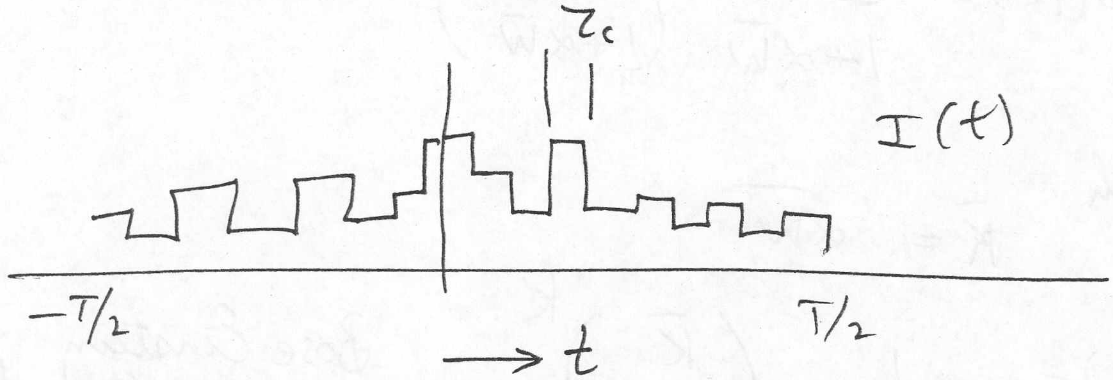
$$\therefore \quad \sigma_K^2 = \bar{K} + \bar{K}^2$$

$$SNR = \frac{\bar{K}}{\sigma_K} = \sqrt{\frac{\bar{K}}{1+\bar{K}}}$$

For $\bar{K} \ll 1$, $SNR \to \sqrt{\bar{K}}$ like laser light

$\bar{K} \gg 1$, $SNR \to 1$!

ex. ~~Integ~~ Thermal light $\quad \Delta t \gg \tau_c$



Over $\Delta t \lesssim \tau_c$, the statistics of $I(t)$ are exponential. We assume that $I(t)$ changes value abruptly ~~for~~ on timescale of Δt.

~~Let~~
$$W = \int_{-T/2}^{T/2} I(t)\,dt \simeq \sum I_i \Delta t$$
$$\simeq \frac{T}{\tau_c} \bar{W}$$

where $\bar{W}$ is the mean integrated intensity over ~~Δt~~ τ_c

We know $\bar{W}$ obeys exponential statistics

The characteristic function of $\bar{W}$ is

$$M_I(\omega) = \frac{1}{1 - j\omega \bar{I}}$$

Thus

$$M_W(\omega) = \left(\frac{1}{1 - j\omega \bar{I}}\right)^m \qquad m = T/\tau_c$$

$$p_W(W) = \frac{\left(\frac{m}{\bar{I}T}\right)^m W^{m-1} \exp\left(-m\frac{W}{\bar{I}T}\right)}{\Gamma(m)} \qquad W \geq 0$$

"gamma prob. density .."

$$\bar{W} = \bar{I}T$$

$$\sigma_W^2 = \frac{(\bar{I}T)^2}{m}$$

$$\therefore \sigma_K^2 = \bar{K} + \alpha^2 \frac{(\bar{I}T)^2}{m}$$

$$= \bar{K} + \frac{\bar{K}^2}{m}$$

$$\sigma_K^2 = \bar{K}\left(1 + \frac{\bar{K}}{m}\right)$$

δ = degeneracy parameter.

When $\delta \ll 1$ . . . Poisson

$\delta \gg 1$ Wave-limit

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$$\therefore \sigma_K^2 = \bar{K} + \alpha^2 \frac{(\bar{I}T)^2}{m}$$

$$= \bar{K} + \frac{\bar{K}^2}{m}$$

$$\sigma_K^2 = \bar{K}\left(1 + \underbrace{\frac{\bar{K}}{m}}_{\delta}\right)$$

δ = degeneracy parameter.

When $\delta \ll 1$ . . . Poisson

$\delta \gg 1$ Wave-limit