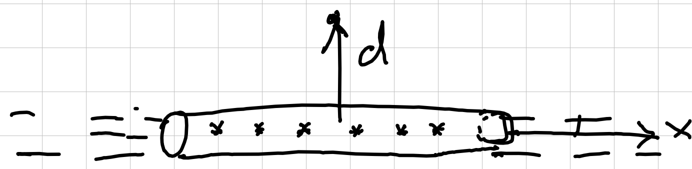
Relativity at work

(inspired by Purcell's Berkeley series E&M book)

I. Consider a long cylinder made of plastic with embedded charges



Let $\lambda_0 =$ linear density of charges (charge per cm)

Frame: K : At point $x=0, y=b$

$\nabla \cdot E = 4\pi\rho \Rightarrow \int E \cdot dA = 4\pi \int \rho \, dv$

$\therefore$

$$E_\perp = \frac{4\pi\lambda_0}{2\pi d} = \frac{2\lambda_0}{d}$$

$E_\parallel = 0 \qquad B_\parallel = 0 \qquad B_\perp = 0$

Frame K':

Transformation equations:

$$E'_{\parallel} = 0 \qquad B'_{\parallel} = 0$$

$$E'_{\perp} = \gamma E_{\perp} \qquad B'_{\perp} = -\gamma \vec{\beta} \times \vec{E}$$

In frame K': $\lambda' = \gamma \lambda_0$ (length contraction)

$$i' = \lambda' v \quad \text{(current)}$$

$$= \gamma \lambda_0 v$$

Electrostatic field:

$$E'_{\perp} = \frac{2\gamma\lambda_0}{d} = \gamma E_{\perp}$$ ✓

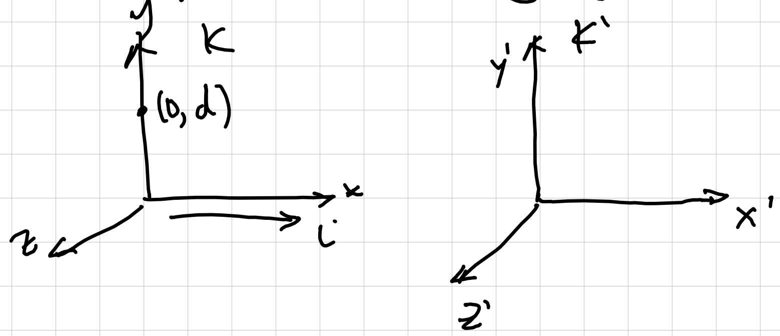
$$\nabla \times B = \frac{4\pi j}{c} + \frac{1}{c}\frac{\partial E}{\partial t} \quad \Rightarrow \quad \oint B' \cdot dl = \frac{4\pi}{c} \int j \cdot dA$$

$$B'_{\perp} = -\frac{4\pi i'}{2\pi d c} = -\frac{2\gamma\lambda_0 v}{dc}$$

$$= -\gamma \frac{2\lambda_0}{d} \beta$$

$$B'_{\perp} = -\gamma \beta E$$ ✓

II Consider a copper wire carrying current, i



In frame K:

$\nabla \cdot E = 4\pi\rho$. The wire is neutral, $\vec{E} = 0$

$$\nabla \times B = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi j}{c}$$

$$\oint B \cdot dl = \frac{4\pi i}{c} \Rightarrow B_\perp = \frac{4\pi i}{2\pi d c} = \frac{2i}{dc}$$

where $d = \sqrt{y^2 + z^2}$ $\qquad B_\parallel = 0$

In frame K' which moves with velocity v

$$E'_\parallel = E_\parallel \qquad B'_\parallel = B_\parallel$$

$$E'_\perp = \gamma(E_\perp + \beta \times B) \qquad B'_\perp = \gamma(B_\perp + \beta \times E)$$

$$\Rightarrow \quad E'_\parallel = 0 \qquad B'_\parallel = 0$$

$$E'_\perp = \gamma \beta \times B \qquad B'_\perp = \gamma B_\perp$$

Now let us look inside the wire

Frame K:

$$\begin{matrix} + & + & + & + & + & + & + \\ - & - & - & - & - & - & - \end{matrix}$$

① Assume for simplicity ions have $+q$ charge

electrons have $-q$ charge

By construction, charge density of electrons equal to charge density of ions, λ_0

net charge: $+\lambda_0 - \lambda_0 = 0$

$\Rightarrow \vec{E} = 0$

② Ions are fixed

③ Electrons move with velocity $v_x = -v$

So, no electrostatic charge in K

$$i = -\lambda_0 v_x = \lambda_0 v_x = \lambda_0 v$$

$$\therefore B_t = \frac{2i}{dc} = \frac{2\lambda_0 v}{dc}$$

tangential

$$= \frac{2\lambda_0 \beta}{d}$$

Now consider frame K' in which electrons are at rest in K':

The electrons are at rest. The ions are moving with velocity $v_x = v$. The charge density of ions is now $\lambda_0' = \gamma \lambda_0$.

$$\text{So } i = \lambda_0' v = \gamma v \lambda_0$$

The magnetic field is tangential (y,z plane)

$$\nabla \times B = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi j}{c}$$

$$\oint B_t' \cdot dl = \frac{4\pi i'}{c}$$

$$B_t' = \frac{4\pi i'}{2\pi d c} = \frac{2\gamma v \lambda_0}{dc} = \frac{2\gamma \lambda_0 \beta}{d}$$

$$B_t' = \gamma B_t \quad \checkmark$$

The electrons are at rest. Thus their charge density is $-\frac{\lambda_0}{\gamma}$ (so that in lab frame it is $-\lambda_0$)

The charge density is

$$-\frac{\lambda_0}{\gamma} + \lambda_0\gamma = \lambda_0\left(\gamma - \frac{1}{\gamma}\right) = \lambda_0\frac{(\gamma^2-1)}{\gamma}$$

$$\gamma^2 = \frac{1}{1-\beta^2} \qquad \gamma^2 - 1 = \frac{\beta^2}{1-\beta^2} = \gamma^2\beta^2$$

The charge density is $\lambda_0\gamma\beta^2$

$$\nabla \cdot E' = 4\pi\rho$$

$$\Rightarrow E'_\perp = \frac{4\pi\lambda_0\gamma\beta^2}{2\pi d}$$

$$= \frac{2\lambda_0\gamma\beta^2}{d}$$

$$E'_\perp = \gamma\beta B_t$$

✓