Recap:

$$U = \gamma \begin{pmatrix} c \\ \vec{v} \end{pmatrix}$$

$$a^{\mu} = \frac{dU^{\mu}}{d\tau} \; ; \qquad p^{\mu} = m U^{\mu}$$

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$

$$F^{\mu} = \frac{q}{c} F^{\mu}{}_{\nu} U^{\nu}$$

$$\vec{F} \cdot \vec{U} = 0$$

$$P = \frac{dW}{dt} \qquad dW = \gamma\, dW' \qquad dt = \gamma\, dt'$$

$$\therefore P' = \frac{dW'}{dt'} = P$$

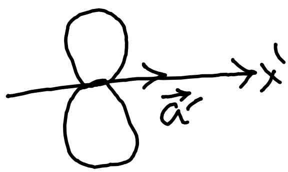
Larmor formula:

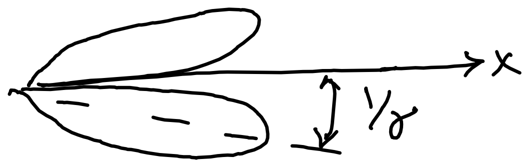
$$P = \frac{2q^2}{3c^3} \underbrace{\vec{a} \cdot \vec{a}}_{\text{four-vector}}$$

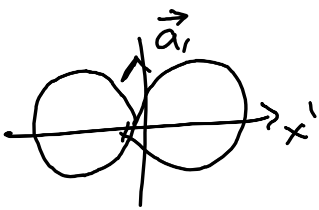
$$a'_{\parallel} = \gamma^3 a_{\parallel}$$

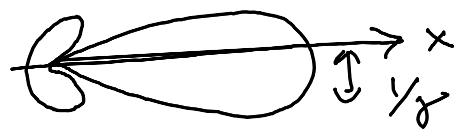
$$a'_{\perp} = \gamma^2 a_{\perp}$$

$$P = \frac{2q^2}{3c^3}\gamma^4\left(a_{\perp}^2 + \gamma^2 a_{\parallel}^2\right)$$

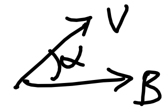


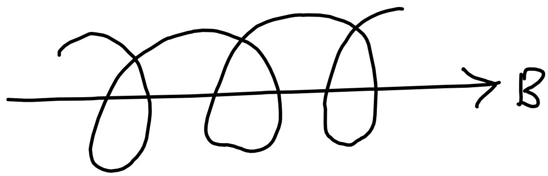






Synchrotron emission:





Force equation:

$$\frac{d}{dt}(\gamma m v) = \frac{q}{c} v \times B$$

$$\frac{d}{dt}(\gamma m c^2) = q\, v \cdot E = 0 \quad \text{since } E = 0$$

Thus $\gamma = \text{constant}$ (v^2 is constant)

$$m\gamma \frac{dv}{dt} = \frac{q}{c} v \times B$$

Separating,

$$\frac{dv_{\parallel}}{dt} = 0 \qquad \text{(coasting along } \vec{B}\text{)}$$

$$\frac{dv_{\perp}}{dt} = \frac{q}{\gamma m c} v_{\perp} \times B \qquad \text{(circle)}$$

The angular rotation rate is

$$\omega_B = \frac{qB}{\gamma m c} \qquad r_g = \frac{v_{\perp}}{\omega_B}$$

The acceleration is $a_{\perp} = \omega_B v_{\perp}$
(it is perpendicular to $v_{\perp}$)

$$\therefore \quad P = \frac{2q^2}{3c^3}\gamma^4 \frac{q^2B^2}{\gamma^2 m^2 c^2} v_\perp^2$$

$$= \frac{2}{3} r_0^2 c \beta_\perp^2 \gamma^2 B^2 \qquad \left[\frac{e^2}{r_0} = mc^2\right]$$

For an isotropic distribution of α, the pitch angle, for a fixed value of β

$$\langle \beta_\perp^2 \rangle = \frac{\beta^2}{4\pi}\int \sin^2\alpha \, d\Omega$$

$$= \frac{2\beta^2}{3}$$

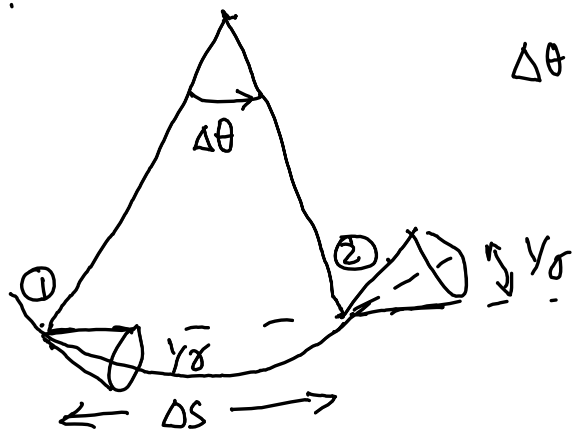
$$P = \frac{4}{3}\sigma_T c \beta^2 \gamma^2 U_B \qquad U_B = \frac{B^2}{8\pi}$$

$$t_{1/2} \simeq \frac{E}{-\left(\frac{dE}{dt}\right)} = \frac{5\times 10^8}{\gamma_0 B_\perp^2}$$

Crab Nebula.

$B = 1\,\text{mG}$ $\qquad t_{1/2} \sim \frac{10^7 \text{ years}}{\gamma_0}$

$\gamma_0 = 10^5$ (for optical)



$$\Delta\theta = \frac{2}{\gamma}$$

R = radius of curvature

$$R = \frac{\Delta s}{\Delta\theta}$$

$$\Delta\theta = \frac{2}{\gamma} \qquad \therefore \Delta s = \frac{2 \cdot R}{\gamma}$$

The radius of curvature of a helix

$x = a\cos(t)$, $y = a\sin(t)$, $z = bt$

$$\frac{1}{R} = \frac{|a|}{a^2+b^2}$$

$$\boxed{\text{Exercise. Show: } R = \frac{v}{\omega_B \sin\alpha}}$$

$$\therefore \quad s = \frac{2v}{\gamma\,\omega_B \sin\alpha}$$

$$t_2 - t_1 = \frac{s}{v} = \frac{2}{\gamma\,\omega_B \sin\alpha}$$

However, pulse emitted at ① travels at speed of light.

Thus $\Delta t^A = \frac{\Delta s}{v} - \frac{\Delta s}{c} = \frac{\Delta s}{v}\left(1 - \frac{v}{c}\right)$

(arrival time difference)

$$\Delta t^A = \frac{2}{\gamma \omega_B \sin\alpha}\left(1 - \frac{v}{c}\right)$$

For relativistic velocities, $\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}}$

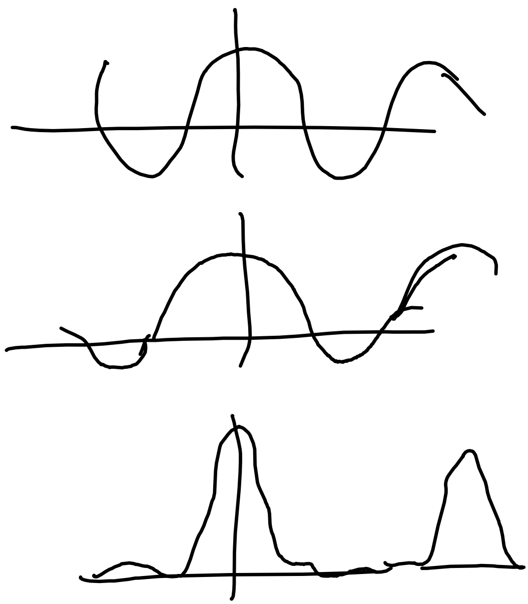
$$\Rightarrow \left(1 - \frac{v}{c}\right) = \frac{1}{2\gamma^2}$$

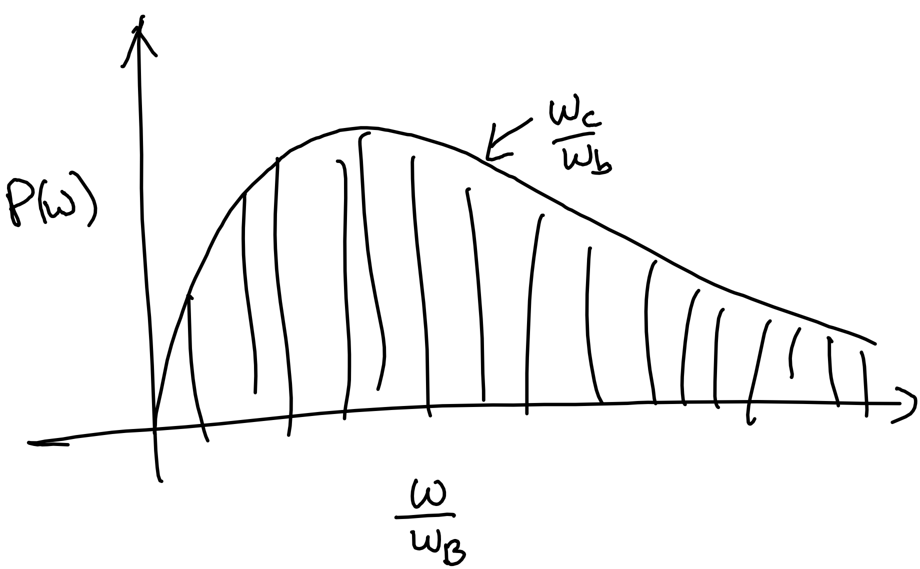
$$\Delta t^A = \left(\gamma^3 \omega_B \sin\alpha\right)^{-1} \propto \frac{1}{\gamma^2}$$

Cyclotron to Synchrotron

$\beta \simeq 0$ cyclotron

as $\beta \uparrow$ electric field





Notice that the electric field of accelerated relativistic electrons is

$$E(t) \propto F(\gamma\theta)$$

We have $\theta \simeq \frac{s}{R}$

$$t \simeq \frac{s}{v}\left(1 - \frac{v}{c}\right)$$

For $\gamma >> 1$, $v = c$, $1 - \frac{v}{c} = \frac{1}{2\gamma^2}$

$$t \propto \frac{s}{\gamma^2} \sim \frac{R\theta}{\gamma^2}$$

but $R = \frac{v}{\omega_B \sin\alpha}$

$$\therefore \quad t \propto \frac{\theta}{\omega_B \sin\alpha\, \gamma^2} \propto \frac{\gamma\theta}{\omega_B \sin\alpha\, \gamma^3}$$

$$\propto \frac{\gamma\theta}{\omega_c}$$

$$\therefore \quad \boxed{\omega_c t \propto \gamma\theta}$$

Define $\omega_c \equiv \frac{3}{2}\gamma^3 \omega_B \sin\alpha \propto \gamma^2$

$$E(t) \propto F(\gamma\theta) \propto g(\omega_c t)$$

$$\tilde{E}(\omega) \propto \int g(\omega_c t) e^{i\omega t} dt$$

$$\propto \int g(\xi) e^{i\frac{\omega}{\omega_c}\xi} d\xi \qquad \xi = \frac{\omega}{\omega_c}$$

$$\Rightarrow P(\omega) = C_1 F\left(\frac{\omega}{\omega_c}\right)$$

$$P = \int_0^\infty P(\omega) d\omega = C_1 \int F\left(\frac{\omega}{\omega_c}\right) d\omega$$

$$= \omega_c C_1 \int_0^\infty F(x) dx \qquad x = \frac{\omega}{\omega_c}$$

but $$P = \frac{2q^4 B^2 \gamma^2 \beta^2 \sin^2\alpha}{3m^2c^3}$$

$$\Rightarrow P(\omega) = \underbrace{\frac{\sqrt{3}}{2\pi}}_{\uparrow \text{ fudge-factor (for later)}} \frac{q^3 B \sin\alpha}{mc^2} F\left(\frac{\omega}{\omega_c}\right)$$

Consider a power law spectrum of energetic electrons

$$N(E)dE = CE^{-p}dE \qquad E_1 < E < E_2$$

$$\Rightarrow \quad N(\gamma)d\gamma = C\gamma^{-p}d\gamma \qquad \gamma_1 < \gamma < \gamma_2$$

$$P_{tot}(\omega) = C\int P(\omega)\,\gamma^{-p}d\gamma$$

$$\propto \int F\left(\frac{\omega}{\omega_c}\right)\gamma^{-p}d\gamma$$

Define $x = \frac{\omega}{\omega_c} = \frac{\omega}{\gamma^2}$ $\qquad \gamma = \left(\frac{\omega}{x}\right)^{\frac{1}{2}}$

$$d\gamma = \omega^{1/2}x^{-1/2}$$

$$\propto \omega^{-\frac{(p-1)}{2}}\int_{x_1}^{x_2} F(x)\,x^{\left(\frac{p-1}{2}\right)}dx$$

For a power-law distribution $\frac{x_2}{x_1} \gtrsim 10$ then for a power-law $\frac{x_2}{x_1}$ may as well be 0 to ∞

$$P_{tot}(\omega) \propto \omega^{-\frac{(p-1)}{2}}$$

Start with the Larmor formula &
bash through

$$\frac{dW_\perp}{d\omega} = \frac{\sqrt{3}\, q^2 \gamma \sin\alpha}{2c}\left[F(x)+G(x)\right]$$

$$\frac{dW_\parallel}{d\omega} = \frac{\sqrt{3}\, q^2 \gamma \sin\alpha}{2c}\left[F(x)-G(x)\right]$$

$$F(x) = x\int_x^\infty K_{5/3}(\xi)\,d\xi$$

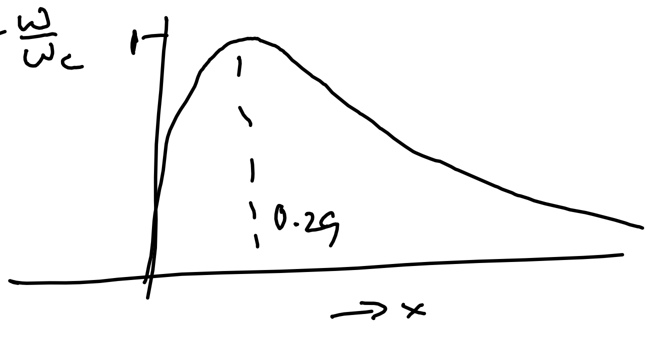
$$G(x) = x\,K_{\frac{2}{3}}(x)$$

$$P(\omega) = \frac{\sqrt{3}\, q^3 B \sin\alpha}{2\pi m c^2}\,F(x)$$

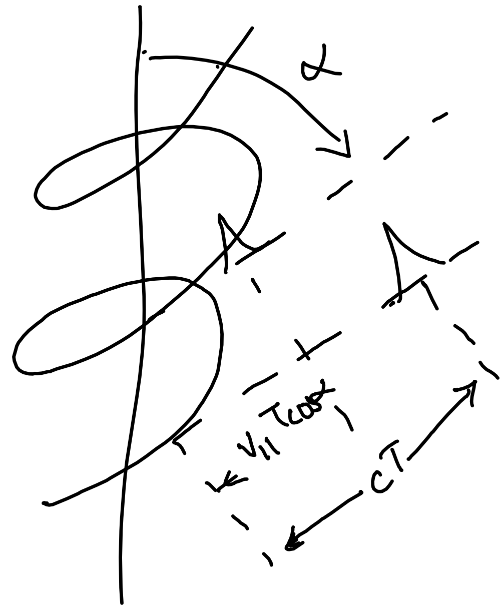
$$F(x) \sim \frac{4\pi}{\sqrt{3}\,\Gamma(\frac{1}{3})}\left(\frac{x}{2}\right)^{1/3} \qquad x \ll 1$$

$$\sim \left(\frac{\pi}{2}\right)^{1/2} e^{-x}\, x^{1/2} \qquad x \gg 1$$

$x = \frac{\omega}{\omega_c}$



Received & Emitted Power:



$$T_A = T\left(1 - \frac{v_{\parallel}}{c}\cos\alpha\right)$$

$$= T\left(1 - \frac{v}{c}\cos^2\alpha\right)$$

$$\simeq T\sin^2\alpha$$

$$= \frac{2\pi}{\omega_B}\sin^2\alpha$$

So the spacing is $\frac{\omega_B}{\sin^2\alpha}$ and not ω_B

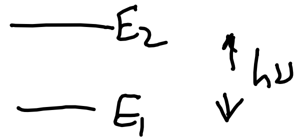
The received power must be obtained by dividing the energy by T_A

$$P_r = \frac{P_e}{\sin^2\alpha} \leftarrow \text{emitted}$$

Synchrotron Self-Absorption:

The absorption coefficient per particle is

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} \left[n(E_1) B_{12} - n(E_2) B_{21} \right] \phi(\nu) \quad (1)$$



$$\phi_{21}(\nu) = \delta(E_2 - E_1 - h\nu)$$

The rate of emission per particle is

$$P(\nu, E_2) = 2\pi P(\omega)$$

$$= h\nu \sum_{E_1} A_{21} \phi_{21}(\nu)$$

$g_1 = g_2$

$$= \frac{2h\nu^3}{c^2} h\nu \sum_{E_1} B_{21} \phi_{21}(\nu) \quad (2)$$

Use (2) to express stimulated absorption

$$-\frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} n(E_2) B_{21} \phi(\nu) =$$

$$\frac{c^2}{8\pi h\nu^3} \sum_{E_2} n(E_2) P(\nu, E_2) ,$$

Likewise, the true absorption can be expressed as
(note $B_{12} = B_{21}$)

$$\frac{+h\nu}{4\pi} \sum_{E_1} \sum_{E_2} n(E_1) B_{12} \Phi_{21}$$

$$= \frac{c^2}{8\pi h \nu^3} \sum_{E_2} n(E_2 - h\nu) P(\nu, E_2)$$

$$\alpha_\nu = \frac{c^2}{8\pi h \nu^3} \sum_{E_2} \left[n(E_2 - h\nu) - n(E_2) \right] P(\nu, E_2)$$

Let $f(p)$ = electron momentum distribution
(assumed isotropic)

$$f(p) d^3p = f(p) \, 4\pi p^2 dp$$

The number of quantum states per unit volume in d^3p is $\frac{2}{h^3} d^3p$

Thus the electron density per quantum state is

$$\frac{h^3}{2} f(p)$$

$$\therefore \quad n(E_2) = \frac{h^3}{2} f(p_2)$$

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\int d^3p_2\left[f(p_1)-f(p_2)\right]P(\nu, E_2) \qquad (A)$$

$$p_1 \to E_1 = E_2 - h\nu$$

For thermal distribution

$$f(p) = K\exp\left(-\frac{E(p)}{kT}\right)$$

$$f(p_1)-f(p_2) = K\left[\exp\left(-\frac{E_2-h\nu}{kT}\right)-\exp\left(-\frac{E_2}{kT}\right)\right]$$

$$= f(p_2)\left[\exp\frac{h\nu}{kT}-1\right]$$

$$(\alpha_\nu)_{\text{Thermal}} = \frac{c^2}{8\pi h\nu^3}\left[\exp\frac{h\nu}{kT}-1\right]\underbrace{\int d^3p_2\, f(p_2)P(\nu, E_2)}_{\substack{\text{total power per volume}\\ \text{per frequency range}\\ =4\pi j_\nu}}$$

$$(\alpha)_{\text{Thermal}} = \frac{j_\nu}{B_\nu(T)}$$

and we have recovered Kirchoff's law

Relativistic particles

$$E = pc$$

$$N(E)\,dE = f(p)\,d^3p \Rightarrow \frac{N(E)}{4\pi\left(\frac{E^2}{c^2}\right)}\,dE = f(p)$$

Use (A) to find

$$\alpha_\nu \propto \int dE\; P(\nu; E)\, E^2 \left[\frac{N(E-h\nu)}{(E-h\nu)^2} - \frac{N(E)}{E^2}\right]$$

$$\frac{\partial}{\partial E}\left(\frac{N(E)}{E^2}\right) h\nu$$

$$\propto \int dE\; P(\nu, E)\, E^2 \frac{\partial}{\partial E}\left[N(E)\right] h\nu \qquad (B)$$

For relativistic particles

$$N(E) \propto E^2 e^{-E/kT}$$

$$\underset{(\text{thermal})}{\alpha_\nu} = \frac{c^2}{8\pi \nu^2 kT} \int N(E)\, P(\nu, E)\, dE = \frac{j_\nu c^2}{2\nu^2 kT}$$

Kirchoff's law in R-J limit

For power-law

$$N(E)\,dE = C E^{-p}\,dE$$

$$-E^2 \frac{d}{dE}\left(\frac{N(E)}{E^2}\right) = (p+2) C E^{-(p+1)}$$

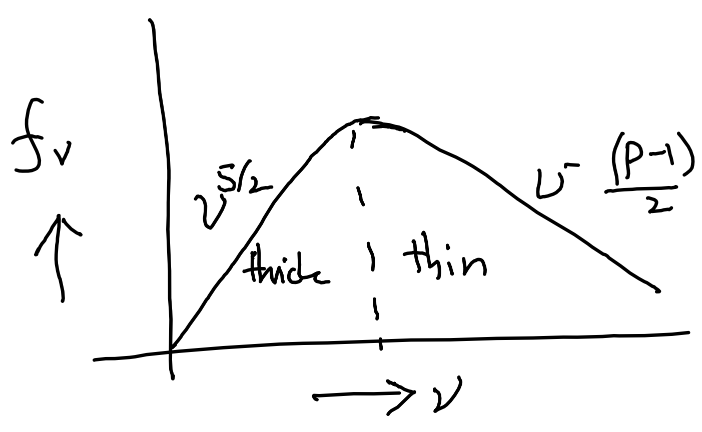
$$= \frac{(p+2) N(E)}{E}$$

$$\alpha_\nu = \frac{(p+2)c^2}{8\pi\nu^2} \int dE\, P(\nu, E) \frac{N(E)}{E}$$

$$\propto B^{\left(\frac{p+2}{2}\right)} \nu^{-\left(\frac{p+4}{2}\right)}$$

Source function:

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{P(\nu)}{4\pi\alpha_\nu} \propto \nu^{5/2}$$



Minimum Energy:

Readhead (1994) noted many radio sources had a limiting brightness temperature, $T \lesssim 10^{11}$ K.

Let $n(E)$ be the energy distribution of relativistic electrons.

The energy in electrons is

$$U_e = \int_{E_1}^{E_2} E\, n(E)\, dE \qquad (1)$$

We simplify by attributing $\nu \leftrightarrow \gamma^2 B$

Thus electron radiating ν satisfies $\gamma^2 B = \text{constant}$

$$\Rightarrow \quad E \propto B^{-1/2} \qquad (2)$$

Next, we have

$$-\frac{dE}{dt} \propto B^2 E^2$$

This loss in energy gives rise to luminosity, L_ν

$$L_\nu \propto B^2 E^{-2}, \qquad L = \int L_\nu \, d\nu$$

Let $n(E) \propto E^{-p}$ for $E_1 < E < E_2$

$$\frac{U_e}{L} \propto \frac{\int_{E_1}^{E_2} E\, n(E)\, dE}{-\int_{E_1}^{E_2} \left(\frac{dE}{dt}\right) n(E)\, dE}$$

$$\propto \frac{E^{2-p} \Big|_{E_1}^{E_2}}{B^2 E^{3-p} \Big|_{E_1}^{E_2}}$$

From (2) we have $E \propto B^{-1/2}$

$$\therefore \frac{U_e}{L} \propto \frac{\left(B^{-1/2}\right)^{2-p}}{B^2 \left(B^{-1/2}\right)^{3-p}} = B^{-3/2}$$

Thus $U_e \propto B^{-3/2}$ for a given L.

The magnetic pressure is $U_B = \frac{B^2}{8\pi}$

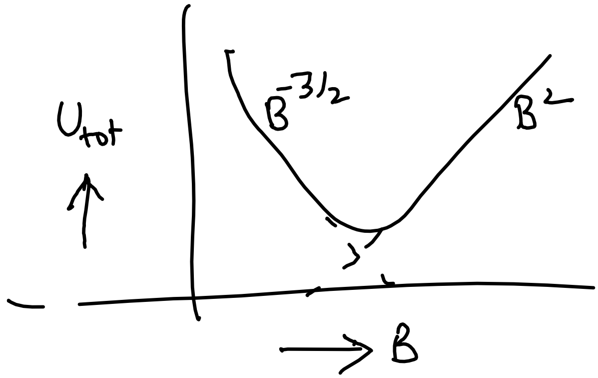
The total energy is

$$U_e + U_p + U_B$$

$\uparrow$ protons

Assume $U_p = \eta U_e$ (shocks)

$$U_{tot} = (1+\eta) U_e + U_B$$



U_{tot} is minimized for

$$\frac{(1+\eta)\, U_e}{U_B} = \frac{4}{3}$$

Apparently, this minimization leads to

(1) "equipartition" (provided $\eta \simeq 1$ and this is not clear)

(2) $T_b \lesssim 10^{11}$ K (Readhead)