Ay 126: Homework 3

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[1] **Energy Difference of Hyperfine levels: A Semi-Classical Calculation.** The goal is to compute the energy splitting of the two hyperfine energy states of H I in ground state. An electron in the ground state of hydrogen has no angular momentum ($1s^1$) and thus does not generate a magnetic field due to orbital motion. However, the spinning electron ($\mathbf{S}$ is magnetized with a a dipole moment

$$\vec{\mu}_e = -g_e \mu_B \mathbf{S} \tag{1}$$

where $g_e \approx 2$ (and set to 2) and $\mu_B = e\hbar/(2m_e)$. Likewise the spinning ($\mathbf{I}$) also has a magnetic moment

$$\vec{\mu}_p = g_p \mu_N \mathbf{I} \tag{2}$$

where $g_p = 5.586$ and $\mu_N = e\hbar/2m_p$ is the nuclear magneton. This spin-spin interaction splits the ground state into two hyperfine levels (upper state with spins aligned, $F = 1$ and lower state with opposing signs, $F = 0$; $\mathbf{F} = \mathbf{S} + \mathbf{I}$).

The interaction energy is $E = -\vec{\mu}_e \cdot \mathbf{B_p}$ where $\mathbf{B}_p$ is the magnetic field generated by the proton dipole. Keeping $\vec{\mu}_e$ fixed in orientation show that this interaction is zero, as long as you exclude the interior of the proton. [It is helpful to draw the field lines of a dipole and qualitatively understand the origin of the stated result].

The electron wave function is given by

$$\psi(r) = \frac{1}{\sqrt{4\pi}} \frac{2}{a_0^{3/2}} \exp(-r/a_0). \tag{3}$$

The electron will spend some time inside the proton. Assuming that interior of the proton is uniformly magnetized, show that the interaction energy (in MKS units) is

$$-E = \frac{4}{3} \frac{\mu_0}{2\pi} \frac{g_e g_p \mu_B \mu_N}{a_0^3} \mathbf{S} \cdot \mathbf{I}. \tag{4}$$

Since the electron and proton spins are 1/2 you find $\mathbf{S} \cdot \mathbf{I} = -3/4, 1/4$. Compare this to the correct answer of 1420 MHz. [15 pts]

[2] **Two Sheets.** The purpose of this exercise is to make you appreciate the difficulty in interpreting H I spectra (and also to give you a simple introduction to self absorption).

For the problem below we assume that there are two sheets of H I with the parameters (Table 1). Each component has a Gaussian velocity distribution distinguished by the velocity wrt Local Standard of Rest (VLSR) and σ_v is the usual Gaussian rms. The other parameters are integrated column density, N_H and T_s, the spin temperature.

Table 1: Two Sheets

Name	VLSR km/s	N_H $10^{20}\,\mathrm{cm}^{-2}$	T_s K	σ_v $\mathrm{km\,s}^{-1}$
A	0	10	50	1
B	1	10	5000	7

Plot the net emission (T_B) and absorption (τ) spectra assuming that (in order of distance from us) (i) A and B and (ii) B and A. [The x-axis should be in km/s]. [10 pts]

[3] *l*-*v* **diagrams.** An assumption that gas clouds are on a circular orbit is reasonable. Consider a line-of-sight (los) starting from the Earth and along Galactic longitude l and latitude $b = 0$. The radial velocity of a cloud is given by

$$v_r = R_0\Big[\Omega(R) - \Omega_0\Big]\sin(l). \tag{5}$$

Here R is the galacto-centric radius of the cloud, R_0 is the radius of the solar circle (the distance from the Sun to the center of Galaxy) and Ω_0 is the local angular speed. $R_0 = 8.5\,\mathrm{kpc}$ and $V_0 = R_0\Omega_0 = 220\,\mathrm{km\,s}^{-1}$. Assume that the rotation curve is flat, that is, $V(R) = V_0$.

1. Derive the result stated in Equation 5. [5 pts]
2. For $l = 45°$ plot the run of v_r (in $\mathrm{km\,s}^{-1}$) as a function of distance from us, d (kpc), all the way to edge of the H I disk (say 20 kpc). [5 pts]
3. Equation 5 offers a ready way to estimate distances to H II regions or giant molecular clouds (in the absence of other distance measures, which is almost always the case). However, there are deviations in the velocity field of the Galaxy due to a triaxial bulge or spiral density waves. Motivated thus we now turn to pedagogy. We choose the following points: (i) the tangent point, (ii) a point on the solar radius (where the los intersects the solar circle; $d \approx 12\,\mathrm{kcp}$) and say at (iii) $d = 20\,\mathrm{kpc}$. Perturb the velocity fields (to keep it simple, just the radial part) by 10 km/s and 30 km/s and derive the corresponding uncertainty in the inferred distances. [5 pts]

[4] **CI and CII FSL radiation CNM, WNM.** This is an open problem. The goal is to get you to appreciate that much of ISM involves detailed atomic physics data (painful but that is how Nature is built). Only when you have the requisite data you can attack the problem. In order to make this problem tractable I suggest that you approach this problem as a group effort with different people researching abundances, depletions and excitation coefficients (collider: H, e^{-1} and if you are keen, He). Then organize a short session in which each student summarizes their findings. With the data at hand each of you is then set to do the rest of the problem.

The goal is to develop a quantitive understanding of the cooling of CNM and WNM.[1]

1. CNM: $n_{\rm H} \approx 50\,{\rm cm}^{-3}$ and $T_k \approx 80\,{\rm K}$.
2. WNM1: $n_{\rm H} \approx 0.5\,{\rm cm}^{-3}$ and $T_k \approx 8,000\,{\rm K}$.
3. WNM2: $n_{\rm H} \approx 2\,{\rm cm}^{-3}$ and $T_k \approx 2,000\,{\rm K}$.

Determine the cooling per H atom for the top six coolants (criteria: abundance, FSL that can be readily excited; and if you wish to be thorough, depletion). [10 pts]

[1] WNM1 is the classical WNM. However, a quarter of the WNM is found to have "lukewarm" temperatures and I simply call this as WNM2.