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Ay 126 Homework #3 Solutions

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(1) First, we show that the interaction energy between the proton dipole magnetic field & the electron magnetic moment is zero, keeping the electron spin $\vec{S}$ in fixed orientation & excluding the interior of the proton.

Magnetic moments
$$\begin{cases} \vec{\mu}_e = -g_e \mu_B \vec{S}, & g_e \approx 2, \ \mu_B = \dfrac{e\hbar}{2m_e} \\ \vec{\mu}_p = g_p \mu_N \vec{I}, & g_p \approx 5.586, \ \mu_N = \dfrac{e\hbar}{2m_p} \end{cases}$$
Bohr magneton; nuclear magneton

Total spin $\vec{F} = \vec{I} + \vec{S}$ (electron & proton contributions)

$$\Rightarrow \hat{H} = -\vec{\mu}_e \cdot \vec{B}_p \qquad \vec{B}_p = \frac{\mu_0}{4\pi r^3}\left[3(\vec{\mu}\cdot\hat{r})\hat{r} - \vec{\mu}\right]$$
proton dipole field

Assume that $\hat{I} = \hat{z}$, s.t. $\hat{S} = \pm\hat{z}$, or the electron & proton spins are parallel/antiparallel

$$\hat{H} = g_e \mu_B \vec{S}\cdot\vec{B}_p = \frac{\mu_0}{4\pi r^3} g_e g_p \mu_B \mu_N \left[3(\hat{\vec{S}}\cdot\hat{r})(\hat{\vec{I}}\cdot\hat{r}) - \hat{\vec{S}}\cdot\hat{\vec{I}}\right]$$

$$(\hat{\vec{S}}\cdot\hat{r}) = \hat{S}(\pm\hat{z}\cdot\hat{r}) = \pm\hat{S}\cos\theta$$
$$(\hat{\vec{I}}\cdot\hat{r}) = \hat{I}(\hat{z}\cdot\hat{r}) = \hat{I}\cos\theta$$
$$\hat{\vec{S}}\cdot\hat{\vec{I}} = \pm\hat{S}\hat{I}$$

$$E = \langle\psi|\hat{H}|\psi\rangle = \frac{\mu_0}{4\pi} g_e g_p \mu_B \mu_N \left\langle\psi\left|\left(\frac{3(\hat{\vec{S}}\cdot\hat{r})(\hat{\vec{I}}\cdot\hat{r}) - \hat{\vec{S}}\cdot\hat{\vec{I}}}{r^3}\right)\right|\psi\right\rangle$$

$|\psi\rangle$ has no angular dependence in the ground state, only radial, & the spin operators acting on the wavefunction will introduce a constant & some angular dependence

$$\therefore \ E \propto \langle 3\cos^2\theta - 1\rangle = \int_0^{2\pi}\int_0^{\pi} 3\cos^2\theta - 1 \sin\theta \, d\theta \, d\phi = \boxed{0} \checkmark$$

So we can neglect the contribution from the dipole field. For subshells with $\ell = 0$, there is a nonzero probability that a state w/ finite electron spin density can be found at the nucleus. As interacting spins approach, the dipole point approximation breaks down — this is the "Fermi contact interaction."

The interior proton field is based on the magnetization of the proton, in the limit of making the "dipole" current loop smaller while keeping the product of the current & area constant,

$$\Rightarrow \vec{B}_p = \frac{2}{3}\mu_0 \vec{\mu}_p \, \delta^3(\vec{r})$$

$$\hat{H} = -\vec{\mu}_e\cdot\vec{B}_p = \frac{2}{3}\mu_0 g_e \mu_B g_p \mu_N (\hat{\vec{S}}\cdot\hat{\vec{I}}) \, \delta^3(\vec{r})$$

$$E = \langle\psi|\hat{H}|\psi\rangle = \frac{2}{3}\mu_0 g_e g_p \mu_B \mu_N \langle\psi|(\hat{\vec{S}}\cdot\hat{\vec{I}}) \, \delta^3(\vec{r})|\psi\rangle$$

Given $\psi(r) = \frac{1}{\sqrt{4\pi}} \frac{2}{a_0^{3/2}} \exp(-r/a_0)$ proton radius

$$\langle\psi|(\hat{\vec{S}}\cdot\hat{\vec{I}})\delta^3(\vec{r})|\psi\rangle = \frac{4}{4\pi}\frac{1}{a_0^3}\int \exp(-2r/a_0)\,\delta^3(\vec{r})\,dV \ \langle\hat{\vec{S}}\cdot\hat{\vec{I}}\rangle$$
spin states of wavefunction

$$= \frac{1}{\pi a_0^3}\langle \hat{\vec{S}} \cdot \hat{\vec{I}} \rangle$$

$$\Rightarrow \boxed{E = \frac{2}{3}\frac{\mu_0}{\pi a_0^3} g_e g_p \mu_B \mu_N \langle \hat{\vec{S}} \cdot \hat{\vec{I}} \rangle}$$

$$\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 = \frac{1}{2}\left(\hat{S}_{net}^2 - \hat{S}_1^2 - \hat{S}_2^2\right) =$$
$$= \frac{1}{2}\left[s_{net}(s_{net}+1) - s_1(s_1+1) - s_2(s_2+1)\right]$$

$$\Rightarrow \langle \hat{\vec{S}} \cdot \hat{\vec{I}} \rangle = \begin{cases} -\frac{3}{4} & s_{net} = 0,\ s_1 = \frac{1}{2},\ s_2 = \frac{1}{2} \quad \text{singlet} \\ \frac{1}{4} & s_{net} = 1,\ s_1 = \frac{1}{2},\ s_2 = \frac{1}{2} \quad \text{triplet} \end{cases}$$

∴ the change in energy between the singlet & triplet configurations

$$\Delta E = \frac{2}{3}\frac{\mu_0}{\pi a_0^3} g_e g_p \mu_B \mu_N \left(\frac{1}{4} + \frac{3}{4}\right) = \frac{1}{6}\frac{\mu_0}{\pi a_0^3} g_e g_p \frac{e^2\hbar^2}{m_e m_p}$$

$$= 9.43 \times 10^{-25}\ \text{J}$$

$$\Delta\nu = \frac{\Delta E}{h} = 1.423\ \text{GHz} = \boxed{1423\ \text{MHz}}$$

which is approximately the correct answer of 1420 MHz.

(2.) First, let us derive the emission & absorption spectra from radiative transfer. For 21 cm HI emission, we have $\nu_{10} = 1420$ MHz, & thus an absorption coefficient of

$$\alpha_\nu = \frac{c^2}{8\pi\nu_{10}^2}\frac{g_1}{g_0} n_0 A_{10}\left[1 - \exp\left(\frac{-h\nu_{10}}{k_B T_S}\right)\right]\phi(\nu)$$

where $A_{10} = \frac{64\pi^4}{3hc^3}\nu_{10}^3 |\mu_B|^2 \approx 2.85 \times 10^{-15}\ s^{-1}$

is the spontaneous emission coefficient for this transition.

Regardless of whether HI is in LTE, we can define the HI spin temperature T_S (analog of molecular excitation temp),

$$\frac{n_1}{n_0} = \frac{g_1}{g_0}\exp\left(\frac{-h\nu_{10}}{k_B T_S}\right)$$

which describes the relative populations of the HI states (Boltzmann distribution)

Since $\frac{g_1}{g_0} = \frac{g_p}{g_e} \approx 3$

& for radio emission $\frac{h\nu_{10}}{k_B T} \ll 1$ is generally a good approximation, $T > T_{CMB}$

we have

$\frac{n_1}{n_0} \approx 3$, where $n_H = n_0 + n_1 \approx 4n_0 \Rightarrow n_0 \approx \frac{1}{4}n_H$

$$\therefore \alpha_\nu \approx \frac{c^2}{8\pi\nu_{10}^2}\frac{3}{4} n_H A_{10}\frac{h\nu_{10}}{k_B T_S}\phi(\nu)$$

This implies that the line-of-sight integrated opacity (optical depth) is

$$\tau_\nu \approx \frac{3c^2}{32\pi\nu_{10}} A_{10} N_H \frac{h}{k_B T_S}\phi(\nu)$$

Let us assume that the line profile is due to thermal Doppler broadening,

$$\phi(v) = \frac{1}{\sqrt{2\pi}\sigma_v}\exp\left(\frac{-(v-v_0)^2}{2\sigma_v^2}\right)$$

in terms of velocity

Since $\phi(v)\,dV = \phi(\nu)\,d\nu$

$$\phi(\nu) = \phi(V)\frac{dV}{d\nu} = \phi(v)\cdot\frac{c}{\nu_{10}}$$

$$\therefore \tau_\nu \approx \frac{3c^3}{32\pi\nu_{10}^2} N_H A_{10} \frac{h}{k_B T_S}\phi(v)$$

$$\approx \frac{3c^3}{32\pi\nu_{10}^2} N_H A_{10}\frac{h}{k_B T_S}\frac{1}{\sqrt{2\pi}\,\sigma_v}\exp\left(\frac{-(v-v_0)^2}{2\sigma_v^2}\right)$$

$$\boxed{\tau_v \approx 2.18\,\frac{N_H}{10^{21}\,cm^{-2}}\,\frac{100\,K}{T_S}\,\frac{km\,s^{-1}}{\sigma_v}\exp\left(\frac{-(v-v_0)^2}{2\sigma_v^2}\right)}$$

Now, from the radiative transfer eq,

$d\tau_\nu = \alpha_\nu\,ds$

$$\Rightarrow \frac{dI_\nu}{d\tau_\nu} = -I_\nu + B_\nu(T_S)$$ for an optically thick gas in thermal equilibrium

Given $I_\nu = \frac{2k_B T_B(\nu)\nu^2}{c^2}$ in the Rayleigh-Jeans regime

$$\Rightarrow \frac{dT_B}{d\tau_\nu} = -T_B + T_S \Rightarrow \boxed{T_B = T_B(0)e^{-\tau_\nu} + T_S(1-e^{-\tau_\nu})}$$

where T_{B0} includes CMB, galactic synchrotron, & some background faint radio sources.

First, let us calculate the optical depth for each source:

$$\tau_v(A) \approx 2.18\left(\frac{10^{21}\,cm^{-2}}{10^{21}\,cm^{-2}}\right)\left(\frac{100\,K}{50\,K}\right)\left(\frac{km\,s^{-1}}{km\,s^{-1}}\right)\exp\left(\frac{-v^2}{2\,km^2 s^{-2}}\right)$$

$$\approx 4.36\exp\left(\frac{-v^2}{2\,km^2 s^{-2}}\right)$$

$$\tau_v(B) \approx 2.18\left(\frac{10^{21}\,cm^{-2}}{10^{21}\,cm^{-2}}\right)\left(\frac{100\,K}{5000\,K}\right)\left(\frac{km\,s^{-1}}{7\,km\,s^{-1}}\right)\exp\left(\frac{-(v-1\,km\,s^{-1})^2}{98\,km^2 s^{-2}}\right)$$

$$\approx (6.2\times10^{-3})\exp\left(\frac{-(v-1\,km\,s^{-1})^2}{98\,km^2 s^{-2}}\right)$$

(i) For A in front of B, we have a net emission spectrum,

$$\Delta T_B^{AB} = (T_S^A - T_B^B)(1-e^{-\tau_A}), \quad \boxed{T_{net}^{AB} = T_B^B e^{-\tau_A} + T_S^A(1-e^{-\tau_A})}$$

$$T_B^B = T_{CMB}e^{-\tau_B} + T_S^B(1-e^{-\tau_B})$$

(ii) For B in front of A, we have

$$\Delta T_B^{BA} = (T_S^B - T_B^A)(1-e^{-\tau_B}), \quad \boxed{T_{net}^{BA} = T_B^A e^{-\tau_B} + T_S^B(1-e^{-\tau_B})}$$

$$T_B^A = T_{CMB}e^{-\tau_A} + T_S^A(1-e^{-\tau_A})$$

Now, for net absorption, we have

$\Delta T_b = T_{src}e^{-\tau}$ where T_{src} corresponds to the illuminating radio source causing the absorption

$$\Rightarrow \tau = \ln\left(\frac{T_{src}}{\Delta T_B}\right)$$

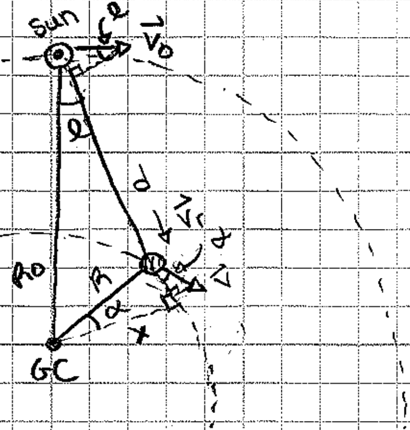
(i) $$\tau_{net}^{AB} = \ln\left(\frac{T_S^B}{\Delta T_B^{AB}}\right)$$

(ii) $$\tau_{net}^{BA} = \ln\left(\frac{T_S^A}{\Delta T_B^{BA}}\right)$$

③ $R_0 = 8.5$ kpc

$V_0 = R_0 \Omega_0 = 220$ km s^{-1}

$V(R) = V_0$ flat rotation curve



1.) To find an expression for the radial velocity of the cloud relative to Earth, we first find the radial velocity of the object along the line of sight,

$V_r = V\cos\alpha$

$\cos\alpha = \frac{x}{R}$ $\qquad \sin l = \frac{x}{R_0} \Rightarrow x = R_0 \sin l$

$\therefore \cos\alpha = \frac{R_0}{R}\sin l, \qquad V_r = V\frac{R_0}{R}\sin l$

Now, we subtract the component that corresponds to the radial velocity along the line of sight of the Earth

$$V_{r,obs} = V\cos\alpha - V_0 \sin l = V\frac{R_0}{R}\sin l - V_0\sin l$$
$$= R_0\left(\frac{V(R)}{R} - \frac{V_0}{R_0}\right)\sin l = \boxed{R_0(\Omega(R) - \Omega_0)\sin l}$$ ✓

2.) Using the Cosine Rule, $R^2 = R_0^2 + d^2 - 2R_0 d\cos l$

with $l = 45°$, $R^2 = R_0^2 + d^2 - \sqrt{2}R_0 d$

$$\therefore \boxed{V_r = \frac{R_0 V_0}{\sqrt{2}}\left(\frac{1}{\sqrt{R_0^2 + d^2 - \sqrt{2}R_0 d}} - \frac{1}{R_0}\right)}$$

see attached plot for d out to 20 kpc

3.) (i) At the tangent pt, $d = R_0 \cos l = \frac{R_0}{\sqrt{2}} \approx 6$ kpc

Let us consider perturbations to the radial velocity, by first inverting the expression to solve for $R(d)$

$$R(d) = \frac{R_0 V_0}{\sqrt{2}V_r + V_0} \Rightarrow \sigma_R = \left|\frac{\partial R(d)}{\partial V_r}\right|\sigma_{Vr} = \frac{\sqrt{2}R(d)}{\sqrt{2}V_r + V_0}\sigma_{Vr}$$

Now, inverting the relationship between d & R, we obtain

$$d(R) = (R^2 - R_0^2 - \sqrt{2}R_0 d)^{1/2}$$

$$\Rightarrow \sigma_d = \left|\frac{\partial d(R)}{\partial R}\right|\sigma_R = \frac{R}{d}\sigma_R = \frac{\sqrt{2}R(d)^2}{d(\sqrt{2}V_r + V_0)}\sigma_{Vr}$$

$d \approx 6$ kpc $\Rightarrow R(6 \text{ kpc}) \approx 6$ kpc

$V_r(6 \text{ kpc}) \approx 64.4$ km s^{-1}

σ_{Vr}: 10 km s^{-1} $\Rightarrow$ $\boxed{\sigma_d = 0.28 \text{ kpc}}$

30 km s^{-1} $\Rightarrow$ $\boxed{\sigma_d = 0.85 \text{ kpc}}$

(ii) $d \approx 12$ kpc $\Rightarrow R(12 \text{ kpc}) \approx 8.49$ kpc

$V_r(12 \text{ kpc}) \approx 0.27$ km s^{-1}

σ_{Vr}: 10 km s^{-1} $\Rightarrow$ $\boxed{\sigma_d = 0.39 \text{ kpc}}$

30 km s^{-1} $\Rightarrow$ $\boxed{\sigma_d = 1.16 \text{ kpc}}$

(iii) $d \approx 20$ kpc, $R(20 \text{ kpc}) \approx 15.2$ kpc

$V_r(20 \text{ kpc}) \approx -68.7$ km s^{-1}

$\boxed{\sigma_d = 1.33 \text{ kpc}}$

$\boxed{\sigma_d = 3.81 \text{ kpc}}$

④ For the CNM, assume $n_H \approx 50\ cm^{-3}$ and $T_K \approx 80K$

The primary coolants in CNM/WNM by abundance (N_x/N_H) are the following:

OI	6.75×10^{-4}
CII	1×10^{-4}
SII	2.24×10^{-5}
SiII	1.91×10^{-5}
FeII	2.82×10^{-6}
CI	1.26×10^{-7}

The cooling rate per H atom is given by the following expression:

$$\Lambda = \frac{n_c n_o}{n_H} K_{o1} \Delta E$$

where the number density of colliders is $n_c = \frac{N_x}{N_H} n_H$

& $K_{o1} = \langle \sigma v \rangle$ is the excitation coefficient

Assume that $\frac{n_e}{n_H} = \begin{cases} 3\times 10^{-4} & \text{in CNM} \\ 0.02 & \text{in WNM} \end{cases}$

Let us consider collisions with H & e^-:

For e^--ion collisions: $\langle \sigma v \rangle_{l \to u} = \frac{8.629 \times 10^{-8}}{\sqrt{T_4}} \frac{\Omega_{ul}}{g_l} e^{-\Delta E / k_B T}$ (1)

For neutral-ion collisions: $\langle \sigma v \rangle = 8.98 \times 10^{-10} Z \left(\frac{\alpha_N}{a_0^3} \right)^{1/2} \left(\frac{m_H}{\mu} \right)^{1/2}$ (2)

For neutral-neutral collisions: $\langle \sigma v \rangle = 1.81 \times 10^{-10} \left(\frac{T}{10^2 K} \right)^{1/2} \left(\frac{m_H}{\mu} \right)^{1/2} \left(\frac{R_1 + R_2}{2 Å} \right)^2$ (3)

Note that the e^--ion collision equation is applicable for e^--neutral collisions (OI).

★ We will go through the example of determining the cooling rate of OI for the CNM:

We will consider the most prominent FSL transition, ${}^3P_2 \to {}^3P_1$ at $63\ \mu m$. This corresponds to $\Delta E = 3.15 \times 10^{-14}$ ergs.
$l \to u$

$n_c = (6.75 \times 10^{-4}) n_H \approx 0.03\ cm^{-3}$

Assuming $g_l = 3$ & $\Omega_{ul} = 0.041\, T_4^{0.69} + 0.064\, T_4^{1.72} \approx 1.48 \times 10^{-3}$ from Draine Table F.3

$\Rightarrow \langle \sigma v \rangle_{l \to u} = 2.75 \times 10^{-11}\ cm^3\ s^{-1}$

Now, for the case that the target is neutral hydrogen (neutral-neutral collisions), we use the temperature-dependent expressions from Draine Table F.6 when available:

for H-OI: ${}^3P_2 - {}^3P_1$: $K_{o1} = 3.57 \times 10^{-10}\, T_4^{\,0.419 - 0.003 \ln T_2} \approx 3.25 \times 10^{-10}\ cm^3\ s^{-1}$

Now, let us determine the associated cooling rates:

e^- collider: $\Lambda = (0.03\ cm^{-3})(3\times10^{-4})(2.75\times10^{-11}\ cm^3\ s^{-1})(3.15\times10^{-14}\ erg)$

$= \boxed{7.8\times10^{-30}\ erg\ s^{-1}\ \text{per H atom}}$

or $\boxed{\Lambda = 3.9\times10^{-28}\ erg\ cm^{-3}\ s^{-1}}$

H collider: $\boxed{\begin{array}{l}\Lambda = 3.07\times10^{-25}\ erg\ s^{-1}\ \text{per H atom} \\ \Lambda = 1.5\times10^{-23}\ erg\ cm^{-3}\ s^{-1}\end{array}}$ $\frac{n_H}{n_H} = 1$

Now, for WNM1: $n_H \sim 0.5\ cm^{-3}$, $T \sim 8000\ K$ $\Rightarrow n_c \approx 3.4\times10^{-4}\ cm^{-3}$

$\Omega_{ul} = 0.079$

e^- collider: $\boxed{\begin{array}{l}\Lambda = 5.36\times10^{-28}\ erg\ s^{-1} \\ 2.7\times10^{-28}\ erg\ cm^{-3}\ s^{-1}\end{array}}$ $\langle\sigma v\rangle = 2.5\times10^{-9}\ cm^3\ s^{-1}$

H collider: $\boxed{\begin{array}{l}\Lambda = 2.25\times10^{-26}\ erg\ s^{-1} \\ 1.1\times10^{-26}\ erg\ cm^{-3}\ s^{-1}\end{array}}$ $\langle\sigma v\rangle = 2.11\times10^{-9}\ cm^3\ s^{-1}$

And for WNM2: $n_H \sim 2\ cm^{-3}$, $T \sim 2000\ K$ $\Rightarrow n_c \approx 1.4\times10^{-3}\ cm^{-3}$

$\Omega_{ul} = 0.018$

e^- collider: $\boxed{\begin{array}{l}\Lambda = 8.3\times10^{-28}\ erg\ s^{-1} \\ 1.7\times10^{-27}\ erg\ cm^{-3}\ s^{-1}\end{array}}$ $\langle\sigma v\rangle = 9.42\times10^{-10}\ cm^3\ s^{-1}$

H collider: $\boxed{\begin{array}{l}\Lambda = 5.4\times10^{-26}\ erg\ s^{-1} \\ 1.1\times10^{-25}\ erg\ cm^{-3}\ s^{-1}\end{array}}$ $\langle\sigma v\rangle = 1.22\times10^{-9}\ cm^3\ s^{-1}$