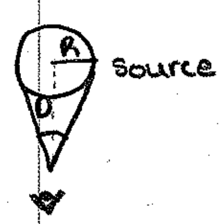
Ay 126 Homework #4 Solutions 5/16/17

(1) 1.1 Assume that the duration of the burst limits the size of the source, s.t. $R \lesssim c\Delta t$ where Δt is the pulse width & c is the speed of light, & R is the radius of the source, assuming non-relativistic flows.

The brightness temperature is defined as the temperature at which a blackbody matches the intensity of the source,

$$I_\nu = B_\nu(T_b) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/k_BT_b}-1} \approx \frac{2h\nu^3}{c^2}\frac{k_BT_b}{h\nu} = \frac{2k_BT_b\nu^2}{c^2}$$

using the Rayleigh-Jeans approximation for radio frequencies $\frac{h\nu}{k_BT} \ll 1$



$$\Rightarrow T_b = \frac{c^2 I_\nu}{2k_B\nu^2} \quad \text{where} \quad I_\nu = \frac{F_\nu}{\Omega}, \quad \Omega = \frac{\pi R^2}{D^2}$$

Let us take $\nu \sim 1.4$ GHz, & $F_\nu \sim 30$ Jy as the peak flux density

$$\Rightarrow T_b = \frac{c^2 F_\nu}{2k_B\nu^2\pi}\left(\frac{D}{R}\right)^2 \quad \therefore \quad T_b \gtrsim \frac{c^2 F_\nu}{2k_B\nu^2\pi}\left(\frac{D^2}{c^2\Delta t^2}\right)$$

$$\boxed{T_b \gtrsim \frac{1}{2\pi k_B}\frac{F_\nu}{\nu^2}\left(\frac{D}{\Delta t}\right)^2}$$

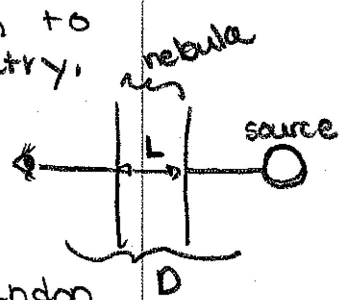
Plugging in values, given that $\Delta t < 5$ ms,

$$T_b \gtrsim (6.7 \times 10^{22}\,\mathrm{K})\left(\frac{D}{1\,\mathrm{kpc}}\right)^2$$

Now, we enforce the incoherent brightness temperature limit, $T_b \simeq 10^{12}$ K, & solve for the distance to the source D,

$$\Rightarrow \boxed{D \lesssim 3.9 \times 10^{-3}\,\mathrm{pc}}$$

If the intervening nebula fills the entire path length to the source, then $DM = Ln_e = Dn_e$ for sheet geometry, s.t. the DM is due entirely to the nebula.



$$\Rightarrow n_e \gtrsim \frac{DM}{D} = \frac{375\,\mathrm{pc\,cm^{-3}}}{3.9 \times 10^{-3}\,\mathrm{pc}} = \boxed{9.6 \times 10^4\,\mathrm{cm^{-3}}}$$

1.2 If we allow for relativistic effects, we can abandon the incoherent brightness temperature limit in the observer's frame, s.t. D does not have to be located so close to us.

First, let us consider the free-free optical depth of the nebula, τ_{ff}. We have the constraint $\tau_{ff} \lesssim 1$, since we are able to observe the emission.

$$\tau_{ff}(\nu) = 2.7\left(\frac{\nu}{\nu_0}\right)^{-2.1}\left(\frac{T_e}{8000\,\mathrm{K}}\right)^{-1.35}\left(\frac{L}{0.01\,\mathrm{pc}}\right)^{-1}$$ from Kulkarni+14

Assuming $T_e \sim 8000\,K$ for the nebula & $\nu \sim \nu_0 = 1.4\,GHz$

$$\tau_{ff} = 2.7\left(\frac{L}{0.01\,pc}\right)^{-1} \lesssim 1$$

$$\Rightarrow \quad L \gtrsim 0.027\,pc$$

Using the constraint $DM = n_e L$, where $DM = 350\,cm^{-3}\,pc$ (accounting for the Galactic contribution of $25\,cm^{-3}\,pc$), we have

$$n_e \lesssim \frac{350\,cm^{-3}\,pc}{0.027\,pc} = \boxed{1.3 \times 10^4\,cm^{-3}}$$

Next, we consider the constraint from Hα radiative recombination, with an upper limit on the surface brightness $I(H\alpha) \lesssim 1$ Rayleigh.

Emission measure, related to the dispersion measure, by $EM = DM^2 L_{pc}^{-1}$ (units of $cm^{-6}\,pc$), can be related to the surface brightness in Rayleighs,

$$\frac{EM}{2.2\,cm^{-6}\,pc} = 0.45\,DM^2 L_{pc}^{-1} \qquad EM = 2.75\,T_4^{0.9}\,I(H\alpha)\,cm^{-6}\,pc$$
$$\approx (2.24\,cm^{-6}\,pc)\,I(H\alpha) \quad \text{for } T \sim 8000K$$

$$\Rightarrow \frac{EM}{2.2\,cm^{-6}\,pc} = I(H\alpha) = 0.45\,DM^2\,L_{pc}^{-1} \lesssim 1 \text{ Rayleigh}$$

$$\therefore\ L \gtrsim 0.45\,DM^2\,pc = 55\,kpc, \Rightarrow \boxed{n_e \lesssim 6.4 \times 10^{-3}\,cm^{-3}}$$

Now, for the final constraint, the fact that we can observe a signal indicates that $\nu > \nu_p$, where $\nu_p = (9\,kHz)\,n_e^{1/2}$ is the plasma frequency.

$$\Rightarrow 1.4\,GHz > (9\,kHz)\,n_e^{1/2} \Rightarrow \boxed{n_e \lesssim 390\,cm^{-3}} \qquad \therefore L \gtrsim 0.89\,pc$$

An additional constraint could be that the thickness of the nebula must be less than the distance to the source, where we restrict the source to the Galaxy and its closest satellites, $L \lesssim D \lesssim 100\,kpc$

$$\Rightarrow \boxed{n_e \gtrsim 3.5 \times 10^{-3}\,cm^{-3}}$$

1.3 Now, we derive the physical properties of the putative nebula, starting with the pressure, assuming an ideal gas & a fully ionized plasma with $n_e = n_H$.

$$\boxed{P_{gas} = n_e k_B T = (1.1 \times 10^{-12}\,erg\,cm^{-3})\left(\frac{n_e}{1\,cm^{-3}}\right)}$$

For the free-free emission flux, we first consider the emissivity, which we calculate from the expression for τ_{ff}.

$$\alpha_{\nu}^{ff} = \tau_{ff}/L = (8.7 \times 10^{-17}\,cm^{-1})\left(\frac{\nu}{\nu_0}\right)^{-2.1}\left(\frac{T_e}{8000\,K}\right)^{-1.35}\left(\frac{L}{0.01\,pc}\right)^{-2}$$

$$\varepsilon_{\nu}^{ff} = 4\pi j_{\nu}^{ff} = 4\pi\,\alpha_{\nu,ff}\,S_{\nu,ff} = 4\pi\,\alpha_{\nu,ff}\,B_{\nu} \approx 4\pi\,\alpha_{\nu}^{ff}\,\frac{2k_B T \nu^2}{c^2}$$

in a thermalized plasma, where the emissivity has units of $erg\,cm^{-3}\,s^{-1}\,Hz^{-1}$

$$\therefore\ \mathcal{E}_\nu^{ff} \approx (6.04 \times 10^{-14}\ \mathrm{erg\ cm^{-3}\ s^{-1}\ Hz^{-1}}) \left(\frac{\nu}{\nu_0}\right)^{-0.1} \left(\frac{T_e}{8000\mathrm{K}}\right)^{-0.35} \left(\frac{L}{0.01\mathrm{pc}}\right)^{-2} \left(\frac{\kappa_{\nu,ff}}{\mathrm{cm^{-1}}}\right)$$

$$\mathcal{E}_\nu^{ff} \approx (5.3 \times 10^{-30}\ \mathrm{erg\ cm^{-3}\ s^{-1}\ Hz^{-1}}) \left(\frac{\nu}{\nu_0}\right)^{-0.1} \left(\frac{T_e}{8000\mathrm{K}}\right)^{-0.35} \left(\frac{L}{0.01\mathrm{pc}}\right)^{-2}$$

To obtain the emission flux, we integrate the emissivity along the path length of the intervening nebula,

$$\boxed{F_\nu^{ff} \approx (1.6 \times 10^{-13}\ \mathrm{erg\ cm^{-2}\ s^{-1}\ Hz^{-1}}) \left(\frac{L}{0.01\mathrm{pc}}\right)^{-1}}$$ assuming $T_e \sim 8000\,\mathrm{K}$ & $\nu \sim \nu_0$

The rate of ionizing photons, assuming that the nebula can be approximated by a cube of size L, is equal to the recombination rate in steady state equilibrium, i.e. $\dot{N}_I = \dot{N}_R$

$\Rightarrow \dot{N}_R = n_e^2 \alpha_B L^3$ for a fully ionized plasma
where $\alpha_B = 8.7 \times 10^{-14}\, T_4^{-0.89}\ \mathrm{cm^3\ s^{-1}} \approx 1.1 \times 10^{-13}\ \mathrm{cm^3\ s^{-1}}$ for $T \sim 8000\,\mathrm{K}$ is the recombination coefficient for case B (optically thick to radiation just above $I_H = 13.6\ \mathrm{eV}$)

$$\Rightarrow \dot{N}_I = \alpha_B\, \mathrm{DM}^2 L = \boxed{(4 \times 10^{47}\ \mathrm{s^{-1}}) \left(\frac{L}{1\mathrm{pc}}\right)}$$

The associated recombination timescale is $t_{rec} = \frac{1}{n_e \alpha_B}$

$$\Rightarrow \boxed{t_{rec} = (0.25\ \mathrm{Myr}) \left(\frac{n_e}{1\ \mathrm{cm^{-3}}}\right)^{-1}}$$

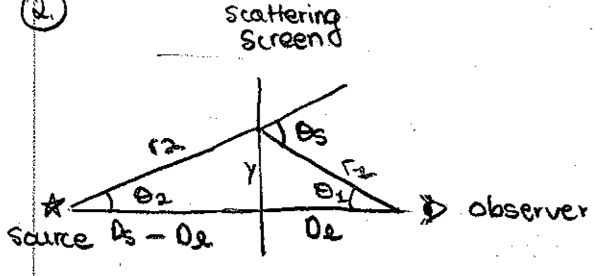
We therefore obtain the following parameters for the nebula,

Constraint	L (pc)	n_e ($\mathrm{cm^{-3}}$)	P_{gas} ($\mathrm{erg\ cm^{-3}}$)	F_ν^{ff} ($\mathrm{erg\ cm^{-2}\ s^{-1}\ Hz^{-1}}$)
$\tau_{ff} \lesssim 1$	0.027^+	$1.3 \times 10^4\ *$	$1.4 \times 10^{-8}\ *$	$5.9 \times 10^{-14}\ *$
$I(H\alpha) \lesssim 1$	$55 \times 10^{3\,+}$	$6.4 \times 10^{-3}\ *$	$7 \times 10^{-15}\ *$	$2.9 \times 10^{-20}\ *$
$\nu_0 > \nu_p$	0.89^+	$390\ *$	$4.3 \times 10^{-10}\ *$	$1.8 \times 10^{-16}\ *$
$D \lesssim 100\,\mathrm{kpc}$	$100 \times 10^3\ *$	3.5×10^{-3}	$3.9 \times 10^{-15}\ +$	$1.6 \times 10^{-20}\ +$

$\dot{N}_I$ ($\mathrm{s^{-1}}$)	t_{rec} (yr)
$1.1 \times 10^{46}\ +$	20^+
$2.2 \times 10^{52}\ +$	$3.9 \times 10^{7}\ +$
$3.6 \times 10^{47}\ +$	640^+
$4 \times 10^{52}\ *$	$7.1 \times 10^{7}\ *$

* = upper limit
\+ = lower limit

(2.)



Generic scattering configuration

$$\theta_s = \theta_1 + \theta_2$$
$$y = \theta_1 D_\ell = \theta_2 (D_s - D_\ell)$$
$$\Rightarrow \theta_1 = \theta_s \frac{D_s - D_\ell}{D_s}$$
$$\theta_2 = \theta_s \frac{D_\ell}{D_s}$$

For interstellar scintillation, wavefronts arriving on different paths will interfere. If the ISM is changing due to relative motion, the interference conditions ~~can~~ vary, leading to intensity variations in radiation. We approximate scattering by the ISM as a 2D "scattering screen" instead of modelling 3D inhomogeneities, with refraction occuring at low frequencies.

The difference in path length that will result in interference is

$$\Delta s = (r_2 + r_1) - D_s$$
$$r_2 = \sqrt{(D_s - D_e)^2 + \theta_2^2 (D_s - D_e)^2} = (D_s - D_e)\sqrt{1 + \theta_2^2} \approx (D_s - D_e)(1 + \tfrac{1}{2}\theta_2^2)$$
$$r_1 = \sqrt{D_e^2 + \theta_1^2 D_e^2} = D_e\sqrt{1 + \theta_1^2} \approx D_e(1 + \tfrac{1}{2}\theta_1^2)$$

using the small angle approximation, assuming cosmological distances for $D_s \sim 1$ Gpc

$$\Rightarrow \Delta s = (r_2 + r_1) - D_s \approx \cancel{D_s} - \cancel{D_e} + (D_s - D_e)\tfrac{1}{2}\theta_2^2 + \cancel{D_e} + \tfrac{1}{2}D_e\theta_1^2 - \cancel{D_s}$$
$$\approx \tfrac{1}{2}(D_s - D_e)\,\theta_s^2\left(\frac{D_e}{D_s}\right)^2 + \tfrac{1}{2}\theta_s^2\left(\frac{D_e}{D_s}\right)^2 (D_s - D_e)\left(\frac{D_s - D_e}{D_e}\right)$$
$$\approx \tfrac{1}{2}\theta_s^2 (D_s - D_e)\left(\frac{D_e}{D_s}\right)^2\left[1 + \frac{D_s - D_e}{D_e}\right] = \frac{\theta_s^2 (D_s - D_e) D_e}{2D_s}$$

This corresponds to a phase difference,

$$\Delta\phi = \frac{2\pi\Delta s}{\lambda} = \frac{\pi\theta_s^2 (D_s - D_e) D_e}{D_s\lambda}$$

For destructive/constructive interference, $\Delta\phi > \pi$

$$\Rightarrow \frac{\pi\theta_s^2 (D_s - D_e) D_e}{D_s\lambda} \gtrsim \pi$$

where θ_s defines the angular size of the scattering disk

$$\therefore \boxed{\theta_s \sim \sqrt{\frac{D_s (c/\nu)}{(D_s - D_e) D_e}}}$$

where λ is the observed signal wavelength.

The associated scale for temporal broadening is therefore

$$\Delta t = \frac{\Delta s}{c} = \boxed{\frac{\theta_s^2 (D_s - D_e) D_e}{2D_s c}}$$

Now, consider a (cosmological) FRB where the temporal broadening was caused by propogation through a turbulent WIM similar to the MW, we have

$$\theta_s = \left(\frac{2D_s}{L}\right)\frac{2\pi e^2}{m_e\omega^2}(\Delta n_e)_L$$

where L is the characteristic lengthscale of variations in n_e

$$\text{6 } \theta_s = \frac{(c/\nu)}{2\pi r_{diff}}$$

(from Draine ch. 11)

where r_{diff} is the diffractive scale.

For Kolmogorov turbulence, we can estimate the size of the diffractive scale (based on lecture),

$$\left(\frac{r_{outer}}{r_{diff}}\right)^{5/3} \sim \underbrace{\Delta\phi^2}_{\text{phase picked up in media}} \sim \left(N_e \frac{c}{\nu}\frac{e^2}{m_e c^2}\right)^2 = \left(\frac{N_e e^2}{m_e c\,\nu}\right)^2$$

where N_e is the electron column density

We can set the outer scale for turbulence by the size of the object, & set $N_e = n_e r_{outer}$

$$\Rightarrow \quad r_{diff} = \left(\frac{m_e c \nu}{n_e e^2}\right)^{6/5} r_{outer}^{-1/5}$$

Assuming typical values for the WIM, $n_e \sim 10^{-3}$ cm^{-3} for a fully ionized plasma, $r_{outer} \sim 1$ kpc for a MW-like galaxy, & $\nu \sim 1.4$ MHz

$$r_{diff} \approx 1.5 \times 10^{9} \text{ cm}$$

$$\Rightarrow \quad D_l \sim \frac{2c\Delta t}{\left(1-\frac{D_l}{D_s}\right)} \left(\frac{2\pi r_{diff}}{(c/\nu)}\right)^2 \quad \text{where} \quad \left(1-\frac{D_l}{D_s}\right) \leq 1$$

$$\therefore \boxed{D_l \gtrsim 36 \text{ pc}}$$

Now, we consider the case of temporal broadening caused by a dense HII region (size ~ 1 pc) surrounding the source, we have $D_s - D_l \simeq 1$ pc & $D_s/D_l \approx 1$

$$\Rightarrow \quad r_{diff} = \frac{c}{2\pi\nu}\sqrt{\frac{(D_s - D_l)}{(c/\nu)}\frac{D_l}{D_s}} \approx \boxed{4.1 \times 10^{10} \text{ cm}}$$

$$n_e = \left(\frac{m_e c \nu}{e^2 r_{outer}}\right)\left(\frac{r_{outer}}{r_{diff}}\right)^{5/6} \approx \boxed{0.16 \text{ cm}^{-3}}$$