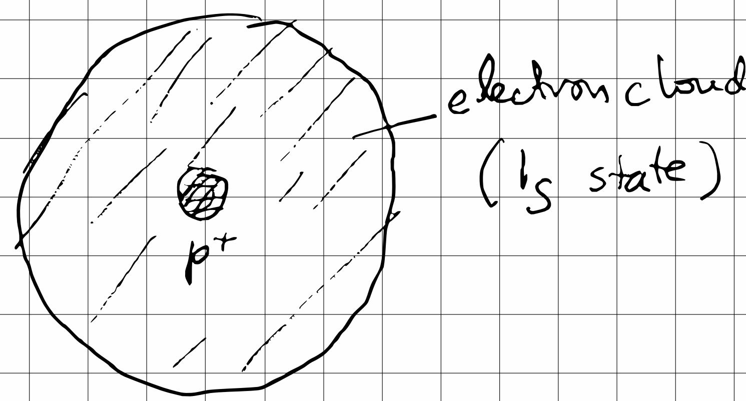
The Hydrogen spin-flip transition.



The electron has a spin $\vec{S}_e$ $\left(\frac{1}{2}\hbar\right)$
The proton has a spin $\vec{S}_p$

The corresponding magnetic momenta are

$$\vec{m}_1 = \gamma_p \vec{S}_p \qquad \vec{m}_2 = \gamma_e \vec{S}_e$$

where γ_p, γ_e are the gyromagnetic ratios.

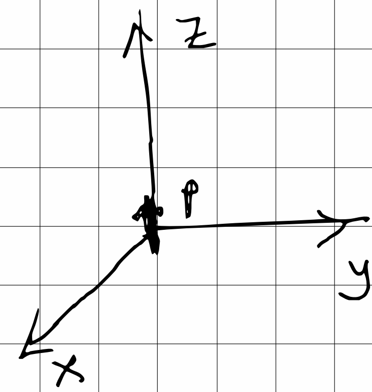
The electron spends some time "inside" the proton and thus we get dipole-dipole interaction. It is this interaction which gives rise to the 21-cm line

Let us first consider electric dipole (which is analogous but not exact) to magnetic dipole.

$$V(r) = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2}$$

$$E(r) = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^3} \left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p} \right]$$

Let us assume that the dipole is oriented along the z-axis



Our goal is to compute the mean electric field of the dipole over a sphere of radius R.

$$\langle E \rangle = \frac{1}{V} \frac{1}{4\pi\varepsilon_0} \int \frac{1}{r^3} \left[3p\cos\theta\, \hat{r} - \hat{p} \right] d^3V$$

$V = \frac{4\pi}{3} R^3 \qquad d^3V = r^2 \sin\theta \, dr\, d\theta\, d\phi$

We have $\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

You can see that the integral over ϕ gets rid of the terms associated with $\hat{x}$ and $\hat{y}$. The mean electric field has a component along the z-axis, that is along the $\vec{p}$ axis

$$\langle E \rangle = \frac{1}{V} \frac{1}{4\pi\varepsilon_0} \vec{p} \int \frac{1}{r^3} \left[3\cos^2\theta - 1 \right] r^2 \sin\theta \, d\theta\, dr$$

Now integrate over θ

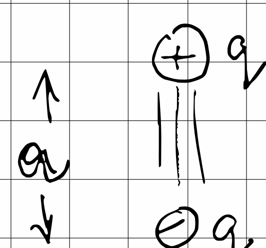
$\therefore$
$$\int_0^{\pi} \left(3\cos^2\theta - 1\right)\sin\theta \, d\theta = 0$$

Thus
$$\langle E \rangle = 0$$

There is a classical theorem in electrostatics

<u>Theorem:</u>
The average electric field over a spherical volume of radius R is
$$\langle E \rangle = -\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}}{R^3}$$
where $\vec{p}$ is the total dipole moment with respect to the center of sphere.

Clearly, something went wrong in our computation of the electric field of a simple dipole.

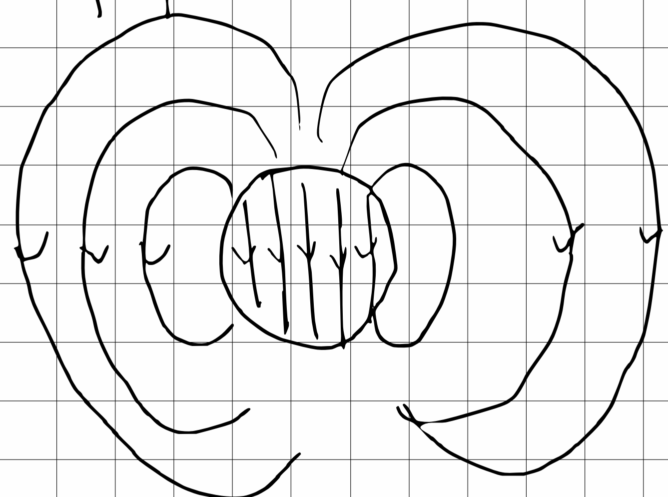


$$\vec{p} = \vec{a}q$$

~~As~~ Dipole: $\vec{p}$ in the limit $a \to 0$ but $q \to \infty$

$\Rightarrow$ There is an enormous field at the center of the dipole

We use instead a uniformly polarized sphere of polarization $\vec{P}$



$$\vec{p} = \frac{4\pi}{3} a^3 \vec{P}$$

The field outside the sphere is a dipolar field.

The field instead is uniform:

$$E_{in}(\vec{r}) = -\frac{1}{4\pi\varepsilon_0} \vec{p} \frac{1}{a^3} \qquad r < a$$

As $a \to 0$ $E_{in}(r)$ blows up.

$$\therefore E_{in}(r) = -\frac{1}{3\varepsilon_0} \vec{p}\, \delta^3(r)$$

The total field is

$$E = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^3} \left[3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p} \right] - \frac{1}{3\varepsilon_0} \vec{p}\, \delta^3(r)$$

with the understanding that the first term is valid for $r > a$ and the second term for $r < a$ and the integration is done in the limit of $a \to 0$

In the same way one can show that the magnetic field of a magnetic dipole is

$$B = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m} \right] + \frac{2}{3} \mu_0 \vec{m}\, \delta^3(\vec{r})$$

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