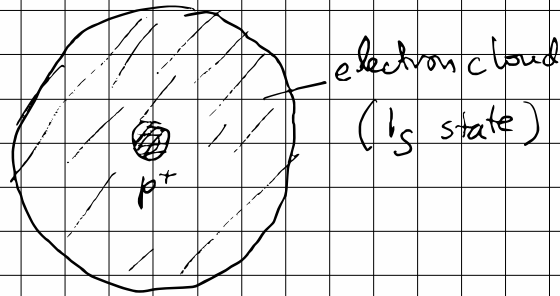


The Hydrogen spin-flip transition.



The electron has a spin \vec{S}_e ($\frac{1}{2}\hbar$)
The proton has a spin \vec{S}_p

The corresponding magnetic moments are

$$\vec{m}_p = \gamma_p \vec{S}_p \quad \vec{m}_e = \gamma_e \vec{S}_e$$

where γ_p, γ_e are the gyromagnetic ratios.

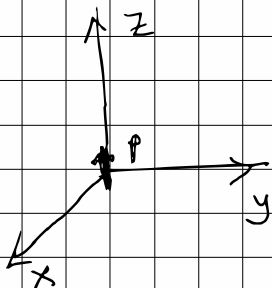
The electron spends some time "inside" the proton and thus we get dipole-dipole interaction. It is this interaction which gives rise to the 21-cm line.

Let us first consider electric dipole (which is analogous but not exact) to magnetic dipole.

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2}$$

$$E(r) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \left[3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p} \right]$$

Let us assume that the dipole is oriented along the z-axis



Our goal is to compute the mean electric field of the dipole over a sphere of radius R .

$$\langle E \rangle = \frac{1}{V} \frac{1}{4\pi\epsilon_0} \int \frac{1}{r^3} [3p \cos\theta \hat{r} - \hat{p}] d^3V$$

$$V = \frac{4\pi}{3} R^3 \quad d^3V = r^2 \sin\theta dr d\theta d\phi$$

We have $\hat{r} = \sin\theta \cos\phi \hat{x} + \sin\theta \sin\phi \hat{y} + \cos\theta \hat{z}$

You can see that the integral over ϕ gets rid of the terms associated with \hat{x} and \hat{y} . The mean electric field has a component along the z-axis that is along the \hat{p} axis

$$\langle E \rangle = \frac{1}{V} \frac{1}{4\pi\epsilon_0} \hat{p} \int \frac{1}{r^3} [3\cos^2\theta - 1] r^2 \sin\theta d\theta dr$$

Now integrate over θ

$$\int_0^{\pi} (3\cos^2\theta - 1) \sin\theta d\theta = 0$$

Thus

$$\langle E \rangle = 0$$

There is a classical theorem in electrostatics

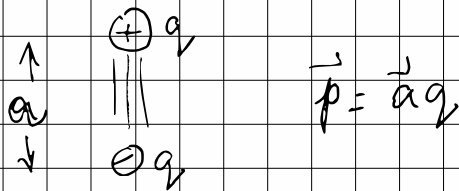
Theorem:

The average electric field over a spherical volume of radius R is

$$\langle E \rangle = -\frac{1}{4\pi\epsilon_0} \frac{\vec{p}}{R^3}$$

where \vec{p} is the total dipole moment with respect to the center of sphere.

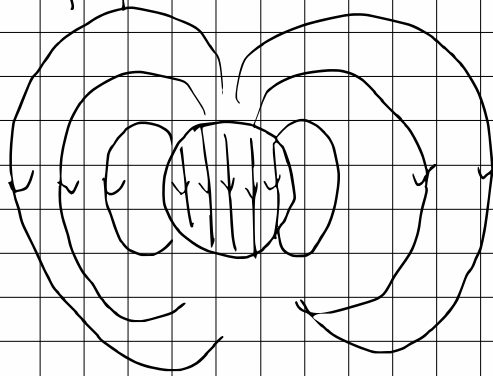
Clearly, something went wrong in our computation of the electric field of a simple dipole.



~~The~~ Dipole: \vec{p} in the limit $a \rightarrow 0$ but $q \rightarrow \infty$

\Rightarrow There is an enormous field at the center of the dipole

We use instead a uniformly polarized sphere of polarization \vec{P}



$$\vec{p} = \frac{4\pi}{3} a^3 \vec{P}$$

The field outside the sphere is a dipole field.
The field inside is uniform:

$$E_{in}(\vec{r}) = -\frac{1}{4\pi\epsilon_0} \vec{P} \frac{1}{a^3} \quad r < a$$

As $a \rightarrow 0$ $E_{in}(r)$ blows up.

$$\therefore E_{in}(r) = -\frac{1}{3\epsilon_0} \vec{P} \delta^3(r)$$

The total field is

$$E = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \left[3(\vec{p} \cdot \hat{r}) \hat{r} - \vec{p} \right] - \frac{1}{3\epsilon_0} \vec{P} \delta^3(r)$$

with the understanding that the first term is valid for $r > a$ and the second term for $r < a$ and the integration is done in the limit of $a \rightarrow 0$

In the same way one can show that the magnetic field of a magnetic dipole is

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[3(\vec{m} \cdot \hat{r}) \hat{r} - \vec{m} \right] + \frac{2}{3} \mu_0 \vec{m} \delta^3(\vec{r})$$