Ay 215: Darwin Instability

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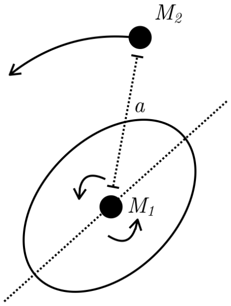
I looked up the original paper by George Darwin (1879). The 179-page paper was daunting. Fortunately, I came across a pedagogical exposition by Mike Lau (2020) and what follows below is a step-by-step derivation.



Consider a planet of mass M_1 orbited by a moon of mass M_2. Say, that the angular velocity of the planet, Ω_1 is smaller than Ω_b, the orbital angular frequency (Figure). If so, as shown in the Figure, the bulge of the planet lags behind the moon. Tidal forces will attempt to increase the rotation of the planet. Increasing the spin of the planet requires energy. Furthermore, doing so will also lead to some dissipation. Gravitational binding energy of the orbit is the source of energy for the two sinks.

Let $\dot{\Omega}_1$ be the rate of change in spin of the planet while that for the orbit is $\dot{\Omega}_b$. The process is stable if $\dot{\Omega}_1 > \dot{\Omega}_b$. If, $\dot{\Omega}_1 < \dot{\Omega}_b$ then the orbit shrinks faster than the planet can be spun up and the result is the moon crashing into the planet.

By construction, $\Omega_1 < \Omega_b$ which means $\Omega_1^{-1} > \Omega_b^{-1}$. The condition for stability is $\dot{\Omega}_1 > \dot{\Omega}_b$. These two inequalities can be combined to yield

$$\frac{\dot{\Omega}_1}{\Omega_1} > \frac{\dot{\Omega}_b}{\Omega_b} \quad \Rightarrow \quad \frac{\dot{\Omega}_1/\Omega_1}{\dot{\Omega}_b/\Omega_b} > 1 \ . \tag{1}$$

Physically, the inequality is readily understood: the timescale for significant increase in Ω_1 which is $\approx \Omega_1/\dot{\Omega}_1$ is shorter than $\Omega_b/\dot{\Omega}_b$, the timescale for equally significant increase of Ω_b.

For a first approximation, the angular momentum of the satellite is ignored ($I_2 = 0$), the orbit is assumed to be circular and the spin of the planet is assumed to be aligned with the orbital angular momentum vector. The total angular momentum is then

$$J = I_1\Omega_1 + \mu a^2\Omega_b$$

where $M = M_1 + M_2$ is the total mass, $\mu = M_1M_2/M$ is the reduced mass, and a is the orbital separation. Since there is no net loss of angular momentum, have

$$\dot{J} = I_1\dot{\Omega}_1 + 2\mu a\dot{a}\Omega_b + \mu a^2\dot{\Omega}_b = 0\ .$$

Kepler's third law is $a^3\Omega_b^2 = GM$. Since there is no loss mass from the system, a log differentiation results in

$$3\frac{\dot{a}}{a} + 2\frac{\dot{\Omega}_b}{\Omega_b} = 0$$

. These two equations can be combined to yield

$$I_1\dot{\Omega}_1 - \frac{1}{3}\mu a^2\dot{\Omega}_b = 0\ .$$

Since, $J_1 = I_1\Omega$ and $J_b = \mu a^2\Omega_b$ we have

$$J_1\frac{\dot{\Omega}_1}{\Omega_1} = \frac{1}{3}J_b\frac{\dot{\Omega}_b}{\Omega_b}\ \Rightarrow\ \frac{\dot{\Omega}_1/\Omega_1}{\dot{\Omega}_b/\Omega_b} = \frac{1}{3}\frac{J_b}{J_1}\ .$$

The condition for stability (Equation1) demands

$$J_b > 3J_1\ .$$

This if the orbital is wide enough,

$$a \geq a_D = \sqrt{\frac{3I_1}{\mu}}\ ,$$

then it is stable.

Hut (1980) included I_2 and showed that the condition for stability for arbitrary eccentricity that the condition for stability is

$$J_b > 3J_1\ \Rightarrow a > a_D = \sqrt{\frac{3(I_1 + I_2)}{\mu}}\ .$$

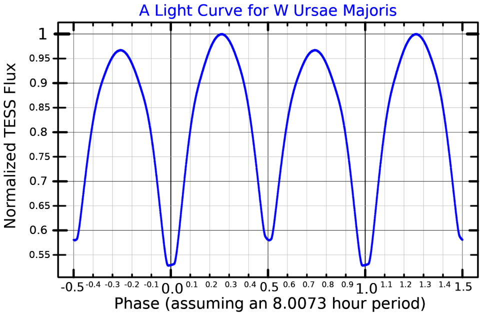


Figure 1: TESS light curve of W UMa.

Contact binaries. W UMa (HD 83950) is a contact binary located some 52 pc away. The orbital period is 0.33 day and the spectral type is F8Vp. $M_1 = 1.14\,M_\odot$, $R_1 = 1.09\,R_\odot$ and $M_1 = 0.55\,M_\odot$ and$R_2 = 0.79\,R_\odot$. Apparently a 12th mag shares a common proper motion with W Uma.

The OGLE project has an excellent atlas for tyros of variable stars. W UMa variables are designated as “EW” stars: “Eclipsing variables of the W Ursae Majoris (W UMa) type are contact binaries, where both stars fill their Roche lobes and therefore are in physical contact with each other. Consequently, the components of the system share a common envelope, leading to the same surface temperature despite their different masses. W UMa stars are characterized by continuous light variations with nearly equal depths of the primary and secondary eclipses. Due to the tidal deformation of the stars, it is impossible to determine the onset and end of the eclipses from the light curve.”

Apparently one in 500 stars in the Galaxy are contact binaries of the W Uma type. Consider a contact binary with mass ratio $q = M_2/M_1$, specifically $q \ll 1$. As before, with this approximation, $I_2 = 0$. We assume that the rotation of the two stars is synchronized. The spin angular momentum of the primary is $J_1 = M_1 k_1^2 R1^2 \Omega_b$ where, k_1 is the dimensionless quantity is the ”radius of gyration”. The condition $J_b > 3J_1$ translates to

$$\frac{a}{R_1} > \left[\frac{3(1+q)}{q}\right]^{1/2} k_1$$

while the Roche lobe radius of star 1 is given by the famed Eggleton approximation:

$$\frac{R_{L1}}{a} = \frac{0.49}{0.6 + q^{2/3}\ln(1+q^{-1/3})}\ .$$

In a contact binary, the stars are filling their Roche lobe, whence, $R_{L1} = R_1$. Combining the above two equations leads to Figure 2. The stable region is clearly marked. $q_{\rm min} \approx k_1^2$ is a good fit to the solid line shown in the left panel Figure 2. Thus, if the primary were $n = 3$ polytrope (approximately a radiative star) then $q_{\rm min} = 0.09$ (Rasio 1995).

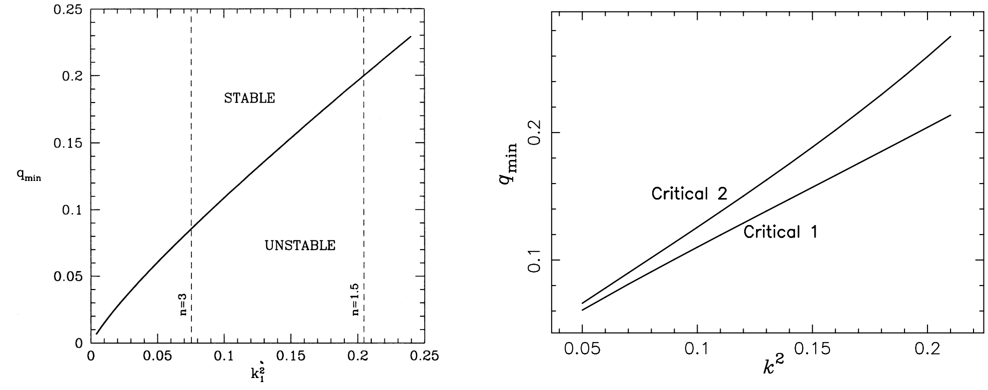


Figure 2: [Right]: The minimum mass ratio for stability $q_{\min}$ for primary in the Roche lobe approximation (solid line). The parameter k_1^2 is the dimensionless gyration radius of the primary. The vertical lines indicate k_1^2 for polytropes of $n = 1.5$ (convective stars, white dwarfs) and $n = 3$ (approximately corresponding to the sun). From Rasio (1995). [Left]: $q_{\min}$ versus k^2 computed by Li & Zhang (2006). The top curve corresponds to the outer Roche lobes and the bottom to the inner Roche lobe.

For the sun, at the present time, $k_1^2 = 0.07$ and so $k_1 = 0.27$. This is in good agreement with the calculations shown in Figure 3.

Li & Zhang (2006) include moment of inertia from the secondary and also account for the "fill" factor (whether the two stars fill the inner Roche lobe – just reaching to L1 or the outer Roche lobe – just reaching to L2). Their stability plot is shown in right panel of Figure 2. Li et al. (2024) report the contact binary TYC 3801-1529-1 has the following properties: $M_1 = 2.096 \pm 0.282\,M_\odot$, $M_2 = 0.075 \pm 0.030\,M_\odot$. The mass ratio is 0.0356 ± 0.0035. This poses a problem to the standard model for contact binaries.

Application to neutron star systems. Cen X-3 was discovered in 1967 in a sounding rocket flight. The source became prominent when Uhuru (1971) discovered pulsations with period of 4.8 s and an orbital period of 2.1 day. The secondary is a neutron star and the primary (Krzemiński's star) is 20.5-$M_\odot$ slightly evolved (O6-7 II-III) star with a radius of $12\,R_\odot$. The extended atmosphere of the primary flows through the L1 point and forms an accretion disk which feeds a magnetized neutron star. Thus, $q \approx 1.5/20.5 = 0.07$. The orbit appears to be decaying, $\dot{P}_b/P_b = 1.8 \times 10^{-6}\,\mathrm{yr}^{-1}$ (Kelley et al. 1983).

According to Zhao & Fuller (2020) "A well-known yet under appreciated aspect of massive stars is that their maximum angular momentum content J_c (assuming rigid rotation) decreases strongly as they evolve off the MS. The reason is that massive stars have very large cores and very tenuous envelopes compared to lower mass stars. As massive stars evolve off the MS, the contraction of the core can outweigh the expansion of the envelope, such

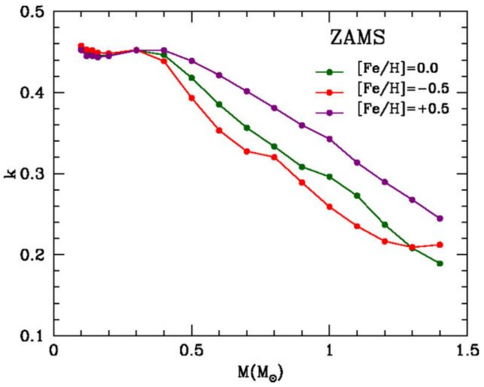


Figure 3: Radius of gyration of main sequence stars as a function of mass (from Wadhwa et al. 2024).

that the radius of gyration can decrease by multiple orders of magnitude.” In this case, let us set $k^2 = 0.1$. If so, the rapid orbital decay can be attributed to the Darwin effect.

Regardless of the real cause for the orbital decay the fact remains that if the measured $\dot{P}$ is secular (and not due to red noise) the end result of Cen X-3 will be a merger between an O star and a neutron star (a Thorne-Żytkow object).

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Mike Lau, *notes* that I found on the web and date June 18, 2020

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Planets in the solar system. The radiation of gyration for a homogeneous sphere is $\sqrt{2/5} \approx 0.632$. The radius of gyration for planets in the solar system is as follows: Mercury (0.588), Venus (0.580), Earth (0.575), Mars (0.605), Jupiter (0.514), Saturn (0.469), Uranus (0.480) and Neptune (0.480). From the the Gravity Recovery and Interior Laboratory (GRAIL mission, $k^2 = 0.392728 \pm 0.00001$ (Williams et al. 2014) which corresponds to $k = 0.627$.