(Developing your own) Handy Formulae

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When you are listening to a talk it is important you compute in real time the effects that the speaker is talking about. If you are an observer then it is important, in addition, to evaluate the feasibility of measuring the effect with existing instruments. These two statements may sound tall and impossible order but with practice the almost impossible becomes possible. What follows below are a collection of tricks and formulae that I have developed over the years that have been helpful to me.

It is my observation that we have difficulty in remembering very large and very small numbers. The numbers easiest to remember are between 0.1 and 10 (or between 1 and 100, if you do not like decimals). The physics related to astronomy – nuclear physics to long-range gravitational force – scales over many orders of magnitude. The way to reconcile our need for small numbers and reality is to through scaling either units or scaling to the characteristics of a familiar source (like the Sun).

> **The Golden Rule.** Redefine units so that the magnitude of the effect is between 0.1 and 10 (or 1 to 100).

There are two strategies to keep numbers small.

1. Bespoke Units. Astronomers in each sub-field have recognized this problem and have defined specialized units specific to their field. The initial definition of magnitude accommodate numbers between 0 and 6. Radio astronomers redefined spectral flux unit (Jansky = $10^{-23}\,\mathrm{erg\,cm^{-2}\,s^{-1}\,Hz^{-1}}$) to suit the first radio surveys (e.g., the flux limit of the 3CR catalog was 10 Jy) whereas solar radio astronomers prefer to use SFU (solar flux unit= 10^4 Jy). The flux density at 10.7-cm wavelength (a standard wavelength in this field) of the sun varies between 50 SFU (solar minimum) and 300 SFU (solar maximum). On the other hand, X-ray astronomers use "Crab" whilst optical astronomers have traditionally used magnitudes. This topic is expanded in §3.

Other examples of bespoke units:Nuclear physicists use "barn" ($10^{-24}\,\mathrm{cm^2}$) for cross-section, atomic physicists use "atomic unit" (au, πa_0^2 where a_0 is the Bohr radius) for cross-section. Astronomers use AU (solar system, planetary astronomy) and pc (Galactic astronomy) for length.

2. Scaling. Sometimes, scaling helps. For instance, it is much easier to remember that the radius of a typical white dwarf is $0.01\,R_\odot$. The solar mass is a good unit for normalizing masses. Recall that the mass of Jupiter is about $10^{-3}\,M_\odot$ while the highest mass brown dwarf is $80\,M_J$. For planetary astronomy, two natural units are the radius of Jupiter and that earth. I like remembering the radius of earth ($R_E \approx 6400\,\mathrm{km}$) because it is the planet we are living and this useful when undertaking long journeys. There are some interesting inter-relations: $R_E = 0.1 R_J$ and $R_J = 0.1\,R_\odot$ where R_J is the radius of Jupiter. Scaling is extensively used in §2

1 Constants that you must remember

In the previous section I emphasized the importance of scaling to make it easier to remember values. However, a serious researcher has no choice but to remember the values given below.

Table 1: Fundamental values that one must remember

c	$3 \times 10^{10}\,\mathrm{cm\,s^{-1}}$
e	4.8×10^{-10} esu
h	$6.6 \times 10^{-26}\,\mathrm{erg\,s}$
N_A	6×10^{23}
m_e	9×10^{-28} g
k_B	1.38×10^{-16} erg/K
σ_T	0.66 barn
σ_{SB}	$5.7 \times 10^{-8}\,\mathrm{erg\,s^{-1}\,cm^{-2}}$
$\lambda_C = h/m_e c$	0.024 Å
$m_e c^2$	511 keV
a_0	0.5 Å
$\alpha = e^2/\hbar c$	1/137
Rydberg	13.6 eV
G	$6.7 \times 10^{-8}\,\mathrm{cm^3\,g^{-1}\,s^{-2}}$
$M_\odot$	2×10^{33} g
$L_\odot$	$4 \times 10^{33}\,\mathrm{erg\,s^{-1}}$
$R_\odot$	7×10^{10} cm (2 light seconds)
AU	1.5×10^{13} cm (500 light seconds)
pc	3×10^{18} cm
year	3×10^{7} s
arcsec	2×10^{5}arcsec/radian
earth radius	6400 km

Electron Volt. Electron volt is used in different contexts. One eV is the energy gained by an electron moving 1 statvolt (which is approximately 300 volts) of potential difference. One eV $= 1.6 \times 10^{-12}$ erg. One eV photon has a wavelength about 1.2 μm. Equating $k_B T$ to 1 eV we find $T \approx 11,000\,K$. The rest mass of a proton is approximately 0.9 GeV (recall that $m_p/m_e \approx 1800$ and the rest mass of the electron is 511 keV).

Time. For time, it is helpful to remember that a day has 86,400 s and that there are 2,000 working[1] hours per years. When computing maximum number of hours for a mission or facility it is useful to remember[2] that a year has 8765 hours.

Angles. There are 2π radians in a circle.

$$1\,\text{radian} = \frac{180^\circ}{\pi} \approx 2 \times 10^5\,\text{arcsec}\ .$$

A sphere has 4π steradians. One steradian is $180/\pi \times 180/\pi$ deg^2 and, many times, I approximate this to 3,300 deg^2. The entire sky has 40,000 deg^2.

Magnitudes. Optical and UV astronomers use the magnitude system:

$$\text{magnitude} = -2.5 \log(\text{flux}) \ + \ \text{zeropoint}.$$

Given this equation you need to have the ability to convert amplitude or factor to log[3] is essential. Memorize the following log values of three prime numbers (second column) and also the last column (in magnitude; it is the second column multiplied by 2.5)

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2  0.3010  0.75
3  0.4771  1.2
7  0.8451  2.1
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You can construct the log of any number to a modest precision from these three numbers. Application: Say, a star flares by a factor of 6. What is the corresponding change in magnitude? Since $6 = 3 \times 2$ the change in magnitude is $-2.5[\log(3) + \log(2)] \approx -2$

Finally, note that $2.5 \log(1 + x) = 2.5 \log(e) \ln(1 + x) \approx 1.08x$ where we have made use of the identity $\ln(1x+) \approx x$ form $x \ll 1$. Say, the source of interest is 19.5 ± 0.1 mag. What is the corresponding fractional uncertainty in the flux. The answer is $0.1 \times 1.08 \approx 10\%$.

Inflation. Let p be the per annum (per year) interest rate. A capital sum increases by $f = (1 + p)^n$ where n is the number of years. Question: what time scales does the capital double? We need $f = 2 = (1 + p)^n$. Thus, $n = \ln(2)/\ln(1 + p) \approx 0.7/p$.

[1] 52 weeks in a year. Assume minimal 2 weeks vacation. Typical work week is 40 hours. Thus the total is $50 \times 40 = 2000$ hours. Very useful to figure out someone's yearly salary given an hour rate.

[2] A year consisting of 365.25 days has 8766 hours but it is easier to remember 8765 than 8766.

[3] I use log for base 10 and ln for natural log

$$\text{doubling time} = \frac{70}{P}\ \text{year}$$

where $P = 100p$ is the interest rate in percentage.

2 Nifty Formulae

I call this as "pre-computation". It helps to have nifty formula so that you can rapidly estimate signal levels in real time (e.g., as during a colloquium).

2.1 Atomic Physics

There is a (bad) tradition of specifying wavelengths in air[4] if between 0.2 μm and 2 μ. In fact, this is the default with ASD/NIST. Fortunately, astronomers are now getting used to quoting wavelengths in vacuum. You must remember these wavelengths: 6563 Å (Hα; average of air and vacuum!), Lyα=1216 Å (vacuum) and Lyman-limit=912 Å.

Hydrogen atom. The Rydberg is binding energy of the hydrogen atom in the ground state. Thus, 1 Rydberg $= -\mathrm{e}^2/(2\mathrm{a}_0)$ where a_0 is the Bohr radius. The velocity in the first orbit is $v = \alpha c$ where $\alpha = e^2/\hbar c$ is the fine structure constant. Thus, 1 Rydberg=$1/2 m_e c^2 \alpha^2$, also.

The following discussion is useful for X-ray astronomy. For hydrogen-like ions the total binding energy is the sum of the potential ($U = -Ze^2/r$) and kinetic ($K = 1/2 m_e v^2 = 1/2 l^2/r^2$) energies where $l = mvr$ is the orbital angular momentum which is quantized. We know $U = -2K$. Thus, the orbital radius scales as Z^{-1} which, given quantization of l, means that $v \propto Z^1$. The energy levels should scale as Z^2 relative to energy. Lyα of Fe^{+25} (Fe XXVI) is then at $3/4 \times 26^2 \times 13.6\,\mathrm{eV}$=6.9 keV.

Oscillator Strength. The absorption oscillator strength, f_{lu} (here l is the lower state and u is the upper state) is related to the A-coefficient,

$$g_u A_{ul} = \frac{8\pi^2 e^2 \nu_{ul}^2}{m_e c^3} g_l f_{lu}.$$

The oscillator strength is formally the absorption cross-section integrated over the line:

$$\int_0^\infty \sigma_\nu d\nu = \frac{\pi e^2}{m_e c} f_{lu}.$$

It is useful to remember $\pi e^2/(m_e c) = 0.0265\,\mathrm{cm}^2\,\mathrm{Hz}$.

[4] the velocity of light in air is smaller than c and so the wavelength, $\lambda = v/\nu$ is smaller by the refractive index at that wavelength

Allowed transitions have oscillator strengths close to unity. For example, the oscillator strengths of some famous permitted lines: Lyα (0.416), Hα (0.614) and Ca H & K (0.345, 0.697). Forbidden transitions have very small oscillator strengths. For example, the famous [OII] doublet has oscillator strengths of $3 \times 10^{-13}, 0.9 \times 10^{-13}$.

Black-body. The black-body intensity formula (Planck function) is

$$B_\nu(T) = \frac{2h\nu^3}{c^2 \exp(h\nu/k_BT - 1}$$

I find it easier to remember it this way:

$$B_\nu(T) = 2 \cdot k^2 \cdot h\nu \cdot \frac{1}{\exp(h\nu/k_BT - 1)}$$

where $k = 1/\lambda$. The factors are 2 for polarization, phase space ($4\pi k^2$ with 4π going to LHS for intensity), photon energy and photon occupation number. From classical physics we know that the spectral energy density, $u_\nu = 4\pi B_\nu/c$ and so $u = 4\pi/cB(T)$ where $B(T) = \int B_\nu d\nu$.

Wein's displacement law: the blackbody peaks $\lambda_p T \approx 0.3\,\mathrm{cm\,K}$ where λ_p is the peak of $B(\lambda)$.

The Stefan-Boltzmann constant is defined by the flux that emerges from a blackbody surface:

$$F = \int B_\nu d\nu d\omega = \pi B(T) = \sigma_{SB} T^4 = \pi B(T)$$

Next ,the average energy of photon is $2.7k_BT$ while the median photon has an energy of only $1.24\,k_BT$. The photon density of a black body is given by $n_\gamma = u(\nu)d\nu/(h\nu) \propto T^3$ and is best remembered as about $n_\gamma = 400\,\mathrm{cm}^{-3}$ for the $T = 2.72\,\mathrm{K}$ CMB. It is useful to remember $u = 1\,\mathrm{erg\,cm}^{-3}$ for T=3.8 K. By happy circumstances the energy density of star light in the solar vicinity is also about $1\,\mathrm{eV/cm}^3$. A magnetic field with an intensity of $6\,\mu$G has a comparable energy density.

Line Width. The thermal 1-D line width of particles with mass, m, is $\sigma = \sqrt{k_BT/m}$. For hydrogen atom, $\sigma = 0.9(T/121\,\mathrm{K})^{1/2}\,\mathrm{km\,s}^{-1}$. For the 21-cm, $1\,\mathrm{km\,s}^{-1}$ corresponds to about 5 kHz. Next, consider Hα (6563 Å). For this wavelength, 1 Å corresponds to $45\,\mathrm{km\,s}^{-1}$.

2.2 Plasma Physics.

Plasma frequency is given by

$$\omega_p = \left(\frac{4\pi n_e e^2}{m_e}\right)^{1/2} \quad \rightarrow \quad \nu_p = 9n_e^{1/2}\,\mathrm{kHz}$$

where n_e is in cm^{-3}.

The gyro-frequency is

$$\omega_B = \frac{eB}{m_e c} \quad \rightarrow \quad \nu_B = 3B\,\mathrm{MHz} \quad \rightarrow \quad E_B = 11B_{12}\,\mathrm{keV}$$

where B is in Gauss. In the interstellar medium, B is typically microGauss. So, the Zeeman splitting is 3 Hz per microGauss. This formula does not work well for cyclotron lines in neutron stars ($B = 10^{12}B_{12}$ Gauss) and for that reason I provide another formula.

Finally, in both cases, note that for relativistic plasma, the electron mass increases and so the formula works by $m_e \rightarrow \gamma m_e$ where γ is the Lorentz factor.

2.3 Stars

The gravitational radius of a star is

$$r_g = \frac{GM}{c^2} = 1.5\Big(\frac{M}{M_\odot}\Big)\,\mathrm{km}$$

and the Schwarzschild radius, $r_S = 2r_g$. The Eddington luminosity which assumes the opacity is produced by electrons in a hydrogen envelope is given by

$$L_{\mathrm{Edd}} = \frac{4\pi GMc(m_p + m_e)}{\sigma_T} = 1.3 \times 10^{38}\left(\frac{M}{M_\odot}\right)\mathrm{erg\,s^{-1}}\ .$$

The escape velocity for star is given by $1/2 v_{\mathrm{esc}}^2 = GM/R$. For the sun, we have

$$v_{\mathrm{esc}}(\odot) = \sqrt{\frac{2GM_\odot}{R_\odot}} = 617\Big(\frac{M}{M_\odot}\Big)^{1/2}\Big(\frac{R}{R_\odot}\Big)^{-1/2}\mathrm{km\,s^{-1}}\ .$$

It turns out that on the main sequence, roughly, $M \propto R$. Thus, the escape velocity of stars on the main sequence is the same as that for the sun. Jupiter has a mass of $10^{-3}\,M_\odot$ and a radius of $10^{-1}\,R_\odot$. Thus, the escape velocity for Jupiter is $62\,\mathrm{km\,s^{-1}}$. A $1\,M_\odot$ white dwarf has a radius which is approximately, $10^{-2}\,R_\odot$ and thus it's escape velocity is ten times that of the sun.

The mean density A star moving at 1 km/s for one million years traverses 1 pc. Thus, a star with a tangential velocity of v_t (km/s) and located at a distance d (parsec) has a proper motion (pm) of

$$\mathrm{pm} = 0.2\Big(\frac{v_t}{d}\Big)\ \mathrm{arcsec/year}.$$

Kepler's third law is

$$\omega^2 = \frac{GM}{a^3}$$

where $\omega = 2\pi/P$ is the orbital angular frequency, a is the semi-major axis and $M = M_1 + M_2$, the sum of the masses of the two objects. Note that the RHS is proportional to the mean density of mass in the binary system. It is better to remember this formula by rescaling

$$\left(\frac{P}{\mathrm{yr}}\right)^2 = \left(\frac{M}{M_\odot}\right)\left(\frac{a}{\mathrm{AU}}\right)^3$$

B. Paczynski (1981) showed that for binary system in which one star fills it Roche lobe (mass, M_2) and the other is a compact star (e.g., cataclysmic variables), the orbital period is related to the density of the Roche-filling star:

$$P(\mathrm{hr}) = 8.85\left(\frac{M_2}{R_2^3}\right)^{-1/2}$$

where M_2 and R_2 are in solar units.

2.4 Earth & Other Planets

We live on earth and it is essential that we know basic parameters of own planet. We approximate the equatorial radius of earth to $a = 6400\,\mathrm{km}$. The earth is flattened at the equator. The ratio of the polar radius to the equatorial radius differ from unity by 3 ppt (parts per thousand).

The mass of earth is is 3 ppm of the mass of sun ($M_E/M_\odot = 3\times10^{-6}$). The escape velocity for earth is $11.2\,\mathrm{km\,s^{-1}}$.

The speed of sound at sea level is $343\,\mathrm{m\,s^{-1}}$. One atmosphere of pressure is about $10^5\,\mathrm{dyne\,cm^{-2}}$ (and equivalently, $10^5\,\mathrm{erg\,cm^{-3}}$ or, more practically, 14.7 psi (pounds per square inch). The ideal pressure in your bicycle tyre depends on what kind of bike you have. I use a simple bike with moderately thick tyres and set the pressure to 32 psi – the same as that for car. Racing bikes have pressures of 80 psi.

One astronomical unit (AU) is about 500 light seconds. The round-trip time for communications between earth and moon is about 2.5 seconds.

The radius of the sun is approximately equal to 2 light seconds. The angular radius of the sun is $2R_\odot/\mathrm{AU}$ which is 4 light seconds (ls) divided by 500 ls or about 0.5°. The angular diameter of the moon ranges between $30'$ and $34'$. It is a nice coincidence that these two are matched.

The earth with a density of $5.5\,\mathrm{gm\,cm^{-3}}$ is the most dense planet in the solar system. The density of the Jupiter is $1.4\,\mathrm{g\,cm^{-3}}$ which is the same as that of the sun. Densities of other notable objects: Mercury ($5.4\,\mathrm{g\,cm^{-3}}$), Mars ($3.9\,\mathrm{g\,cm^{-2}}$) and moon ($3.3\,\mathrm{g\,cm^{-2}}$).

2.5 The Galaxy

2.6 Miscellaneous

3 Flux Units

The standard unit for flux and intensity is Jansky and Jansky/ster with the base as frequency (f_ν, I_ν). Optical astronomers are likely to use f_λ and I_λ with λ in Å whilst mid-IR astronomers would prefer the same but in microns.

However, a more natural unit, especially for intensity (surface brightness) at optical wavelengths is Rayleigh which is

$$1\,\mathrm{Rayleigh} = \frac{10^6}{4\pi}\,\mathrm{phot\,cm^{-2}\,s^{-1}\,ster^{-1}}\ .$$

favored by aeronomers and astronomers studying optical nebulae. Usually Rayleigh is used for intensity that is integrated over a spectral line.

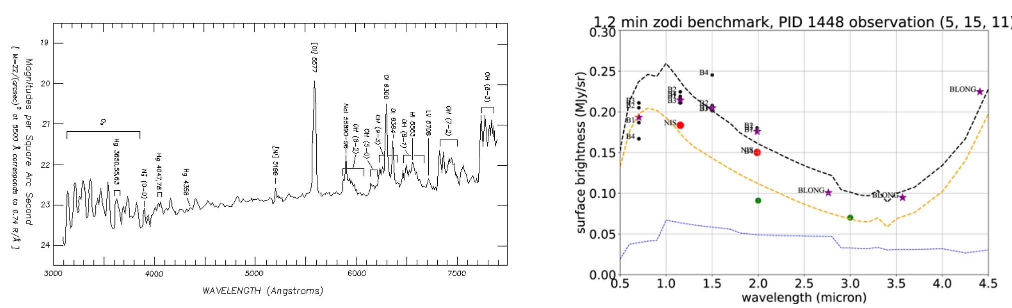

Figure 1: (Left): The sky brightness at Mauna Kea (new moon, clear). CHFT website. (Rihgt): The sky brightness as seen by JWST (Rigby et al. 20240.

It is quite common to specify sky brightness as magnitude per square arcsecond or Mega Jansky per steradian (see Figure 1). Let I_ν carry the unit MJy/ster. It is, many times, useful to convert this to Rayleigh. The intensity in a bandwidth $\Delta\nu$ is $I_\nu\Delta\nu$ and is $I_\nu\Delta\nu/(h\nu)$, but in photon units.

$$I_\nu\frac{\Delta\nu}{h\nu} = \frac{I_\nu}{h}\frac{1}{\mathcal{R}} \approx 189 I_\nu\,\mathcal{R}_3^{-1}\,\mathrm{Rayleigh}$$

where $\mathcal{R} = \Delta\nu/\nu$, $\mathcal{R}_3 = \mathcal{R}/10^3$ and it is understand that I_ν in the right-most term of RHS is in MJy/ster.