FREQUENCY DOMAIN CONCEPTS

The Fourier transform operation take time series and transform them to the frequency domain. Many data processing algorithms can be implemented very well in frequency domain and that is the reason for the popularity of Fourier transforms. For example, consider a sinusoidal wave of frequency f_0. In the frequency domain it can be represented by two δ functions, at $\pm f_0$. The description in the frequency domain is more concise than that in the time domain. Apart from this, there are a number of physical phenonmena in which Fourier transform play an essential role:

1. *Antennas.* The far-field pattern (i.e. Fraunhoffer diffraction) is the Fourier transform of the distribution of the currents in the aperture plane i.e. the illumination pattern.

2. *Optics.* A Fourier transform relation exists between the light amplitude distribution at the back and the front focal planes of a converging lens. For example, consider a point source at infinity. At the front focal plane we have a plane wavefront and at the back focal plane the image is formed i.e. a δ function. The argument can be extended for more complicated sources by considering sources to be made of many point sources each of which give rise to a plane wavefront (with tilts). Thus a lens can be conceptually thought of a Fourier transforming device.

3. *Physics.* The wavefunctions of $\vec{x}$ and $\vec{p}$ are related by a Fourier transform.

4. *Interferometry.* The basis of astronomical interferometric imaging is the van Cittert-Zernike theorem which states that the measured visibility function is the Fourier transform of the object brightness distribution.

5. *Probability.* We have earlier proved that the probability density function of the sum of two independent variables is the convolution of the probability density functions of the two variables. In the frequency domain, the characteristic function of the sum variable is the product of the characteristic functions of the two variables. Clearly, the relation is much simpler in the frequency domain rather than the time domain.

Linear Systems

Most of the standard signal processing devices are linear systems eg. filters, spectrometers, gratings, cameras, interferometers etc. Linear systems satisfy the following general criteria:

Linearity. Let $x(t)$ be an input time varying signal and let $y(t)$ the corresponding output from a linear system i.e.

$$y(t) = \mathcal{H}[x(t)]$$

where $\mathcal{H}$ is the linear operator symbolizing the action performed by the system on the input signal. Then linearity demands that

$$\begin{aligned}\mathcal{H}[x_1(t) + x_2(t)] &= \mathcal{H}[x_1(t)] + \mathcal{H}[x_2(t)] \\ &= y_1(t) + y_2(t).\end{aligned}$$

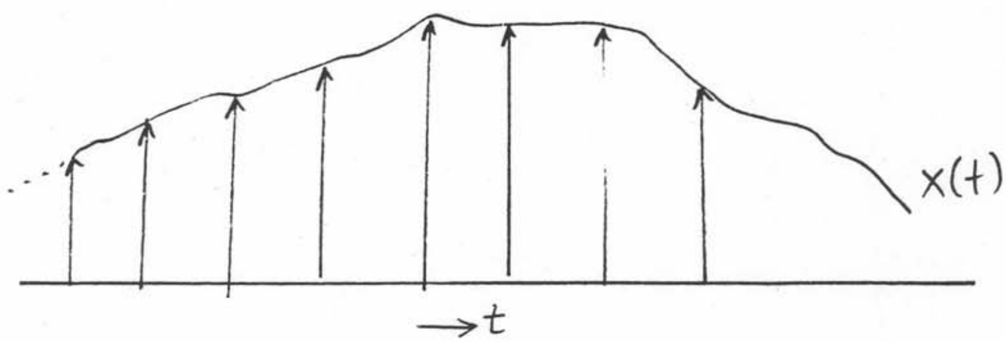
Time-Invariant. A system is time-invariant if a delayed input produces a delayed output i.e.

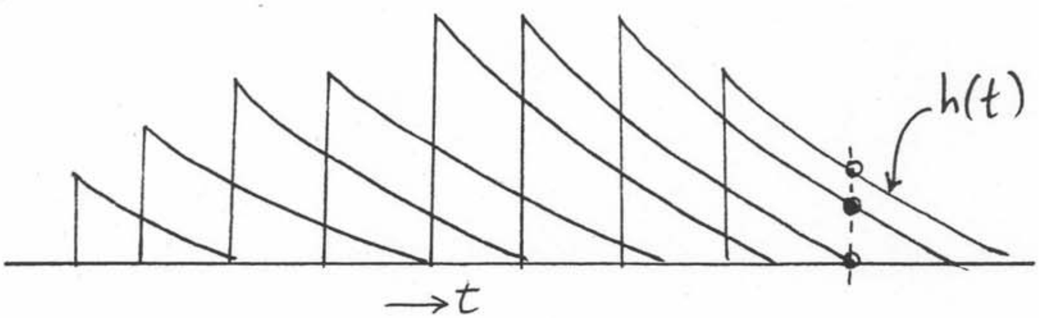
$$y(t - t_0) = \mathcal{H}[x(t - t_0)].$$

Impulse Response Function. All linear systems can be characterized by an impulse response function, $h(t)$,

$$h(t) \equiv \mathcal{H}[\delta(t)]$$

i.e. $h(t)$ is the response of the linear system to a δ- function input. Causality demands $h(t) = 0$ for $t < 0$. Consider an arbitrary input, $x(t)$.





It is quite easy to see that the output $y(t)$ is

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau = \int h(\tau)\, x(t-\tau)\, d\tau$$

i.e. the output of a linear system is a convolution of the impulse response function and the input

$$y(t) = x(t) * h(t).$$

If you think about this, the above equation is quite amazing. The output for *any* input can be predicted, provided you know $h(t)$.

Fourier transforms simply the above equation even one step further. A convolution in the time domain is equalivalent to a multiplication in the frequency domain (this is the famous convolution theorem and is the heart of many astronomical instruments) i.e.

$$\mathcal{F}(y(t)) = \mathcal{F}[x(t) * h(t)]$$
$$= \mathcal{F}(x(t)) \cdot \mathcal{F}(h(t)). \quad F(x(t)) \cdot F(h(t))$$

Thus in the frequency domain, all linear systems can be thought as filters!

In the above discussion we have assumed that the input is a time series. However, the input could well be something else. For example, consider a telescope. The response of a δ-function i.e. a point source, is the so-called Point Spread Function (PSF). The image obtained by a telescope is the convolution between the PSF and the true image. In the Fourier domain, the telescope can be thought of as filter which lets in spatial frequencies no higher than its diameter. Thus the images obtained by a telescope cannot be much sharper than λ/D. However, in practice, the atmospher corrupts the cosmic signals and the pictures obtained are rarely at the diffraction limit of the telescope. The atmosphere acts as another filter – filtering away most of the spatial frequencies above 10 cm wavelengths. (in optical)

FOURIER TRANSFORM

Recommended Books:

The Fast Fourier Transform by E. O. Brigham (Prentice-Hall). I highly recommend this book. A must for any serious observer. Clear and informative.

The Fourier Transform and its Application by R. Bracewell (McGraw-Hill). This is a classic. Particularly good at explaining concepts and peppered with examples. I believe it is now out of print (it is pre-FFT days!).

The Fourier Integral and its Applications by A. Papoulis (McGraw-Hill). A complete compendium. Thorough and slightly more mathematical than the previous two books.

My notes will follow the notation of Brigham's book on the FFT.

$$H(f) = \int_{-\infty}^{+\infty} h(t) e^{-2\pi j f t} dt \qquad \text{Forward Transform}$$

$$g(t) = \int_{-\infty}^{+\infty} G(f) e^{+2\pi j f t} df \qquad \text{Inverse Transform}$$

Consider a signal $h(t)$ and let $H(f)$ be its transform. Let $h'(t)$ be the inverse transform of $H(f)$. Below we show that $h(t)$ is equal to $h'(t)$.

$$h'(t) = \int_{-\infty}^{+\infty} H(f') e^{2\pi j f' t} df'$$
$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(t') e^{-2\pi j f' t'} e^{+2\pi j f' t} df' dt'$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(t) e^{2\pi j f'(t-t')} df' dt'.$$

Since

$$\int_{-\infty}^{+\infty} e^{2\pi j f'(t-t')} df' = \delta(t - t')$$

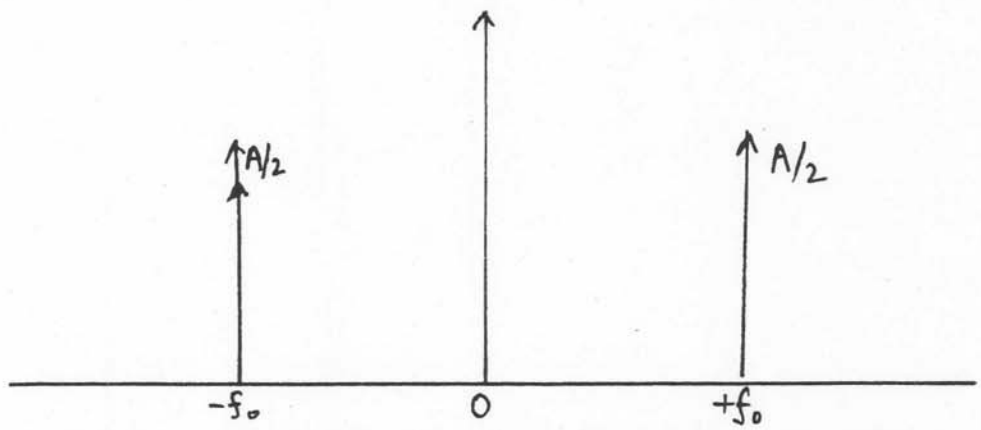
we have

$$h'(t) = h(t).$$

This equality fails if $h(t)$ has discontinuities. However, if h(t) is defined to be $[h(t^+) + h(t^-)]/2$ then the inversion holds.

Consider the Fourier transform of a cosine signal, $h(t) = A\cos(2\pi f_0 t)$. The Fourier transform is

$$H(f) = \frac{A}{2}\delta(f - f_0) + \frac{A}{2}\delta(f + f_0).$$



Note that the amplitude is split between positive and negative frequencies. You may find it difficult to think in terms of negative frequencies but negative frequencies are as real as positive frequencies. This is a simple outcome of the fact that our input signal is real and the Fourier transform is a complex transform and in order to keep the signal representation real we need both positive and negative frequencies. It is best to always think of Fourier transforms in terms of both positive and negative frequencies.

The Fourier transform of a sinusoidal signal, $h(t) = A\sin(2\pi f_0 t)$ is

$$H(f) = \frac{-jA}{2}\delta(f - f_0) + \frac{jA}{2}\delta(f + f_0).$$

Note here that the positive and negative frequency components have opposite phases. This behaviour can be generalized: when the input signal is symmetric about 0 then the positive and negative frequency components are real and equal in magnitude and when the input signal is anti-symmetric about 0 then the positive and negative frequency components are imaginary and conjugate of each other.

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from Brigham "the Fast Fourier Transform"

2

THE FOURIER TRANSFORM

A principal analysis tool in many of today's scientific challenges is the Fourier transform. Possibly the most well-known application of this mathematical technique is the analysis of linear time-invariant systems. But as emphasized in Chapter 1, the Fourier transform is essentially a universal problem solving technique. Its importance is based on the fundamental property that one can examine a particular relationship from an entirely different viewpoint. Simultaneous visualization of a function and its Fourier transform is often the key to successful problem solving.

2-1 THE FOURIER INTEGRAL

The Fourier integral is defined by the expression

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}\,dt \qquad (2\text{-}1)$$

If the integral exists for every value of the parameter f then Eq. (2-1) defines $H(f)$, the Fourier transform of $h(t)$. Typically $h(t)$ is termed a function of the variable time and $H(f)$ is termed a function of the variable frequency. We will use this terminology throughout the book; ***t* is time** and ***f* is frequency**. Further, a lower case symbol will represent a function of time; the Fourier transform of this time function will be represented by the same upper case symbol as a function of frequency.

In general the Fourier transform is a complex quantity:

$$H(f) = R(f) + jI(f) = |H(f)|\,e^{j\theta(f)} \qquad (2\text{-}2)$$

where $R(f)$ is the real part of the Fourier transform,

1.1-7

$I(f)$ is the imaginary part of the Fourier transform,
$|H(f)|$ is the *amplitude* or *Fourier spectrum* of $h(t)$ and is given by $\sqrt{R^2(f) + I^2(f)}$,
$\theta(f)$ is the *phase angle* of the Fourier transform and is given by $\tan^{-1}[I(f)/R(f)]$.

EXAMPLE 2-1

To illustrate the various defining terms of the Fourier transform consider the function of time

$$h(t) = \beta e^{-\alpha t} \quad t > 0$$
$$= 0 \quad t < 0 \tag{2-3}$$

From Eq. (2-1)

$$H(f) = \int_0^\infty \beta e^{-\alpha t} e^{-j2\pi ft}\,dt = \beta \int_0^\infty e^{-(\alpha + j2\pi f)t}\,dt$$
$$= \frac{-\beta}{\alpha + j2\pi f} e^{-(\alpha + j2\pi f)t}\Big|_0^\infty = \frac{\beta}{\alpha + j2\pi f}$$
$$= \frac{\beta\alpha}{\alpha^2 + (2\pi f)^2} - j\frac{2\pi f\beta}{\alpha^2 + (2\pi f)^2}$$
$$= \frac{\beta}{\sqrt{\alpha^2 + (2\pi f)^2}} e^{j\tan^{-1}[-2\pi f/\alpha]} \tag{2-4}$$

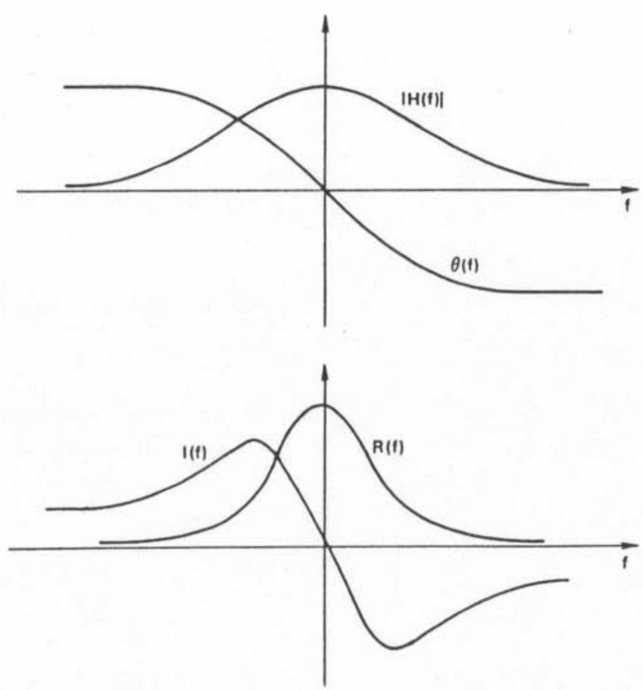

Figure 2-1. Real, imaginary, magnitude, and phase presenta-

Hence

$$R(f) = \frac{\beta\alpha}{\alpha^2 + (2\pi f)^2}$$
$$I(f) = \frac{-2\pi f\beta}{\alpha^2 + (2\pi f)^2}$$
$$|H(f)| = \frac{\beta}{\sqrt{\alpha^2 + (2\pi f)^2}}$$
$$\theta(f) = \tan^{-1}\left[\frac{-2\pi f}{\alpha}\right]$$

Each of these functions is plotted in Fig. 2-1 to illustrate the various forms of Fourier transform presentation.

2-2 THE INVERSE FOURIER TRANSFORM

The inverse Fourier transform is defined as

$$h(t) = \int_{-\infty}^{\infty} H(f)e^{j2\pi ft}\,df \tag{2-5}$$

Inversion transformation (2-5) allows the determination of a function of time from its Fourier transform. If the functions $h(t)$ and $H(f)$ are related by Eqs. (2-1) and (2-5), the two functions are termed a *Fourier transform pair*, and we indicate this relationship by the notation

$$h(t) \Longleftrightarrow H(f) \tag{2-6}$$

EXAMPLE 2-2

Consider the frequency function determined in the previous example

$$H(f) = \frac{\beta}{\alpha + j2\pi f} = \frac{\beta\alpha}{\alpha^2 + (2\pi f)^2} - j\frac{2\pi f\beta}{\alpha^2 + (2\pi f)^2}$$

From Eq. (2-5)

$$h(t) = \int_{-\infty}^{\infty}\left[\frac{\beta\alpha}{\alpha^2 + (2\pi f)^2} - j\frac{2\pi f\beta}{\alpha^2 + (2\pi f)^2}\right]e^{j2\pi ft}\,df$$

Since $e^{j2\pi ft} = \cos(2\pi ft) + j\sin(2\pi ft)$, then

$$h(t) = \int_{-\infty}^{\infty}\left[\frac{\beta\alpha\cos(2\pi ft)}{\alpha^2 + (2\pi f)^2} + \frac{2\pi f\beta\sin(2\pi ft)}{\alpha^2 + (2\pi f)^2}\right]df$$
$$+ j\int_{-\infty}^{\infty}\left[\frac{\beta\alpha\sin(2\pi ft)}{\alpha^2 + (2\pi f)^2} - \frac{2\pi f\beta\cos(2\pi ft)}{\alpha^2 + (2\pi f)^2}\right]df \tag{2-7}$$

The second integral of Eq. (2-7) is zero since each integrand term is an odd function. This point is clarified by examination of Fig. 2-2; the first integrand term in the second integral of Eq. (2-7) is illustrated. Note that the function is odd; that is, $g(t) = -g(-t)$. Consequently, the area under the function from $-f_0$ to f_0 is zero. Therefore, in the limit as f_0 approaches infinity, the integral of the function remains zero; the infinite integral of any odd function is zero.

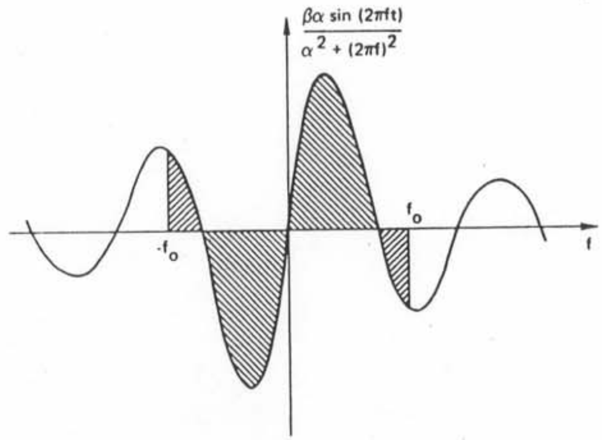

Figure 2-2. Integration of an odd function.

Eq. (2-7) becomes

$$h(t) = \frac{\beta\alpha}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{\cos(2\pi tf)}{(\alpha/2\pi)^2+f^2}df + \frac{2\pi\beta}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{f\sin(2\pi tf)}{(\alpha/2\pi)^2+f^2}df \qquad (2\text{-}8)$$

From a standard table of integrals [4]:

$$\int_{-\infty}^{\infty}\frac{\cos ax}{b^2+x^2}dx = \frac{\pi}{b}e^{-ab} \qquad a>0$$

$$\int_{-\infty}^{\infty}\frac{x\sin ax}{b^2+x^2}dx = \pi e^{-ab} \qquad a>0$$

Hence Eq. (2-8) can be written as

$$h(t) = \frac{\beta\alpha}{(2\pi)^2}\left[\frac{\pi}{(\alpha/2\pi)}e^{-(2\pi t)(\alpha/2\pi)}\right] + \frac{2\pi\beta}{(2\pi)^2}[\pi e^{-(2\pi t)(\alpha/2\pi)}]$$
$$= \frac{\beta}{2}e^{-\alpha t} + \frac{\beta}{2}e^{-\alpha t} = \beta e^{-\alpha t} \qquad t>0 \qquad (2\text{-}9)$$

The time function

$$h(t) = \beta e^{-\alpha t} \qquad t>0$$

and the frequency function

$$H(f) = \frac{\beta}{\alpha + j(2\pi f)}$$

are related by both Eqs. (2-1) and (2-5) and hence are a Fourier transform pair;

$$\beta e^{-\alpha t} \quad t>0 \quad \Longleftrightarrow \quad \frac{\beta}{\alpha + j(2\pi f)} \qquad (2\text{-}10)$$

2-3 EXISTENCE OF THE FOURIER INTEGRAL

To this point we have not considered the validity of Eqs. (2-1) and (2-5); the integral equations have been assumed to be well defined for all functions. In general, for most functions encountered in practical scientific analysis, the Fourier transform and its inverse are well defined. We do not intend to present a highly theoretical discussion of the existence of the Fourier transform but rather to point out conditions for its existence and to give examples of these conditions. Our discussion follows that of Papoulis [5].

Condition 1. If $h(t)$ is integrable in the sense

$$\int_{-\infty}^{\infty}|h(t)|\,dt < \infty \qquad (2\text{-}11)$$

then its Fourier transform $H(f)$ exists and satisfies the inverse Fourier transform (2-5).

It is important to note that Condition 1 is a sufficient but not a necessary condition for existence of a Fourier transform. There are functions which do not satisfy Condition 1 but have a transform satisfying (2-5). This class of functions will be covered by Condition 2.

EXAMPLE 2-3

To illustrate Condition 1 consider the pulse time waveform

$$\begin{aligned} h(t) &= A \qquad &|t| < T_0 \\ &= \frac{A}{2} \qquad &t = \pm T_0 \\ &= 0 \qquad &|t| > T_0 \end{aligned} \qquad (2\text{-}12)$$

which is shown in Fig. 2-3. Equation (2-11) is satisfied for this function; therefore, the Fourier transform exists and is given by

$$H(f) = \int_{-T_0}^{T_0} Ae^{-j2\pi ft}\,dt$$
$$= A\int_{-T_0}^{T_0}\cos(2\pi ft)\,dt - jA\int_{-T_0}^{T_0}\sin(2\pi ft)\,dt$$

The second integral is equal to zero since the integrand is odd;

$$H(f) = \frac{A}{2\pi f}\sin(2\pi ft)\Big|_{-T_0}^{T_0}$$
$$= 2AT_0\frac{\sin(2\pi T_0 f)}{2\pi T_0 f} \qquad (2\text{-}13)$$

Those terms which obviously can be canceled are retained to emphasize the [sin (af)/(af)] characteristic of the Fourier transform of a pulse waveform (Fig. 2-3).

Because this example satisfies Condition 1 then $H(f)$ as given by (2-13) must satisfy Eq. (2-5).

FFT-3

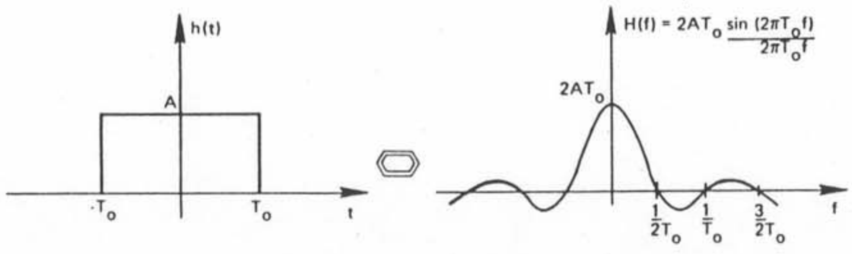

Figure 2-3. Fourier transform of a pulse waveform.

$$h(t) = \int_{-\infty}^{\infty} 2AT_0 \frac{\sin(2\pi T_0 f)}{2\pi T_0 f} e^{j2\pi ft}\, df$$

$$= 2AT_0 \int_{-\infty}^{\infty} \frac{\sin(2\pi T_0 f)}{2\pi T_0 f}[\cos(2\pi ft) + j\sin(2\pi ft)]\, df \qquad (2\text{-}14)$$

The imaginary integrand term is odd; therefore

$$h(t) = \frac{A}{\pi}\int_{-\infty}^{\infty} \frac{\sin(2\pi T_0 f)\cos(2\pi ft)}{f}\, df \qquad (2\text{-}15)$$

From the trigonometric identity

$$\sin(x)\cos(y) = \tfrac{1}{2}[\sin(x+y) + \sin(x-y)] \qquad (2\text{-}16)$$

$h(t)$ becomes

$$h(t) = \frac{A}{2\pi}\int_{-\infty}^{\infty} \frac{\sin[2\pi f(T_0+t)]}{f}\, df + \frac{A}{2\pi}\int_{-\infty}^{\infty} \frac{\sin[2\pi f(T_0-t)]}{f}\, df$$

and can be rewritten as

$$h(t) = A(T_0+t)\int_{-\infty}^{\infty} \frac{\sin[2\pi f(T_0+t)]}{2\pi f(T_0+t)}\, df$$

$$+ A(T_0-t)\int_{-\infty}^{\infty} \frac{\sin[2\pi f(T_0-t)]}{2\pi f(T_0-t)}\, df \qquad (2\text{-}17)$$

Since (| | denotes magnitude or absolute value)

$$\int_{-\infty}^{\infty} \frac{\sin(2\pi ax)}{2\pi ax}\, dx = \frac{1}{2|a|} \qquad (2\text{-}18)$$

then

$$h(t) = \frac{A}{2}\frac{T_0+t}{|T_0+t|} + \frac{A}{2}\frac{T_0-t}{|T_0-t|} \qquad (2\text{-}19)$$

Each term of Eq. (2-19) is illustrated in Fig. 2-4; by inspection these terms add to yield

$$\begin{aligned} h(t) &= A & |t| < T_0 \\ &= \frac{A}{2} & t = \pm T_0 \\ &= 0 & |t| > T_0 \end{aligned} \qquad (2\text{-}20)$$

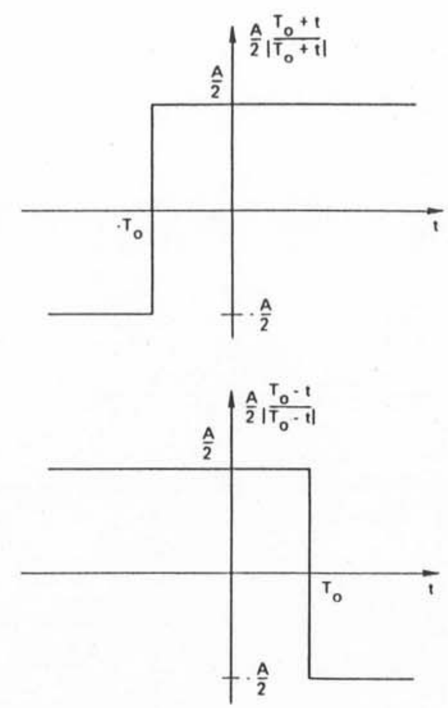

Figure 2-4. Graphical evaluation of Eq. (2-19).

The existence of the Fourier transform and the inverse Fourier transform has been demonstrated for a function satisfying Condition 1. We have established the Fourier transform pair (Fig. 2-3)

$$h(t) = A \quad |t| < T_0 \quad \Longleftrightarrow \quad 2AT_0 \frac{\sin(2\pi T_0 f)}{2\pi T_0 f} \qquad (2\text{-}21)$$

Condition 2. If $h(t) = \beta(t)\sin(2\pi ft + \alpha)$ (f and α are arbitrary constants), if $\beta(t+k) < \beta(t)$, and if for $|t| > \lambda > 0$, the function $h(t)/t$ is absolutely integrable in the sense of Eq. (2-11) then $H(f)$ exists and satisfies the inverse Fourier transform Eq. (2-5).

An important example is the function $[\sin(af)/(af)]$ which does not satisfy the integrability requirements of Condition 1.

EXAMPLE 2-4

Consider the function

$$h(t) = 2Af_0 \frac{\sin(2\pi f_0 t)}{2\pi f_0 t} \qquad (2\text{-}22)$$

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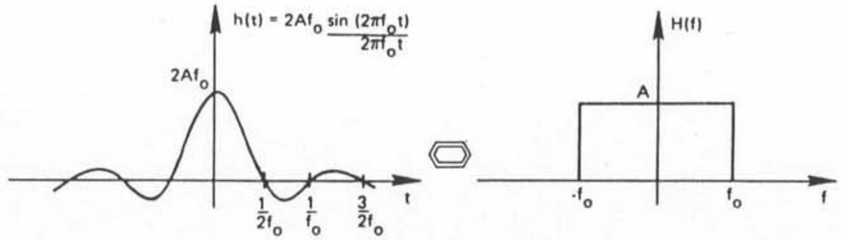

Figure 2-5. Fourier transform of $A \sin(at)/(at)$.

illustrated in Fig. 2-5. From Condition 2 the Fourier transform of $h(t)$ exists and is given by

$$H(f) = \int_{-\infty}^{\infty} 2Af_0 \frac{\sin(2\pi f_0 t)}{2\pi f_0 t} e^{-j2\pi ft}\,dt$$
$$= \frac{A}{\pi}\int_{-\infty}^{\infty} \frac{\sin(2\pi f_0 t)}{t}[\cos(2\pi ft) - j\sin(2\pi ft)]\,dt$$
$$= \frac{A}{\pi}\int_{-\infty}^{\infty} \frac{\sin(2\pi f_0 t)\cos(2\pi ft)}{t}\,dt \qquad (2\text{-}23)$$

The imaginary term integrates to zero since the integrand term is an odd function. Substitution of the trigonometric identity (2-16) gives

$$H(f) = \frac{A}{2\pi}\int_{-\infty}^{\infty} \frac{\sin[2\pi t(f_0 + f)]}{t}\,dt + \frac{A}{2\pi}\int_{-\infty}^{\infty} \frac{\sin[2\pi t(f_0 - f)]}{t}\,dt$$
$$= A(f_0 + f)\int_{-\infty}^{\infty} \frac{\sin[2\pi t(f_0 + f)]}{2\pi t(f_0 + f)}\,dt$$
$$+ A(f_0 - f)\int_{-\infty}^{\infty} \frac{\sin[2\pi t(f_0 - f)]}{2\pi t(f_0 - f)}\,dt \qquad (2\text{-}24)$$

Equation (2-24) is of the same form as Eq. (2-17); identical analysis techniques yield

$$H(f) = A \qquad |f| < f_0$$
$$= \frac{A}{2} \qquad f = \pm f_0$$
$$= 0 \qquad |f| > f_0 \qquad (2\text{-}25)$$

Because this example satisfies Condition 2, $H(f)$ [Eq. (2-25)], must satisfy the inverse Fourier transform relationship (2-5)

$$h(t) = \int_{-f_0}^{f_0} Ae^{j2\pi ft}\,df$$
$$= A\int_{-f_0}^{f_0} \cos(2\pi ft)\,df = A\frac{\sin(2\pi ft)}{2\pi t}\Big|_{-f_0}^{f_0}$$
$$= 2Af_0\frac{\sin(2\pi f_0 t)}{2\pi f_0 t} \qquad (2\text{-}26)$$

By means of Condition 2, the Fourier transform pair

$$2Af_0\frac{\sin(2\pi f_0 t)}{2\pi f_0 t} \quad ⬭ \quad H(f) = A \qquad |f| < f_0 \qquad (2\text{-}27)$$

has been established and is illustrated in Fig. 2-5.

Condition 3. Although not specifically stated, all functions for which Conditions 1 and 2 hold are assumed to be of *bounded variation*; that is, they can be represented by a curve of finite length in any finite time interval. By means of Condition 3 we extend the theory to include singular (impulse) functions.

If $h(t)$ is a periodic or impulse function, then $H(f)$ exists only if one introduces the theory of distributions. Appendix A is an elementary discussion of distribution theory; with the aid of this development the Fourier transform of singular functions can be defined. It is important to develop the Fourier transform of impulse functions because their use greatly simplifies the derivation of many transform pairs.

Impulse function $\delta(t)$ is defined as [Eq. (A-8)]

$$\int_{-\infty}^{\infty} \delta(t - t_0)\,x(t)\,dt = x(t_0) \qquad (2\text{-}28)$$

where $x(t)$ is an arbitrary function continuous at t_0. Application of the definition (2-28) yields straightforwardly the Fourier transform of many important functions.

Example 2-5

Consider the function

$$h(t) = K\delta(t) \qquad (2\text{-}29)$$

The Fourier transform of $h(t)$ is easily derived using the definition (2-28):

$$H(f) = \int_{-\infty}^{\infty} K\delta(t)e^{-j2\pi ft}\,dt = Ke^0 = K \qquad (2\text{-}30)$$

The inverse Fourier transform of $H(f)$ is given by

$$h(t) = \int_{-\infty}^{\infty} [K]e^{j2\pi ft}\,df = \int_{-\infty}^{\infty} K\cos(2\pi ft)\,df + j\int_{-\infty}^{\infty} K\sin(2\pi ft)\,df \qquad (2\text{-}31)$$

Because the integrand of the second integral is an odd function, the integral is zero; the first integral is meaningless unless it is interpreted in the sense of distribution theory. From Eq. (A-21), Eq. (2-31) exists and can be rewritten as

$$h(t) = K\int_{-\infty}^{\infty} e^{j2\pi ft}\,df = K\int_{-\infty}^{\infty} \cos(2\pi ft)\,df = K\,\delta(t) \qquad (2\text{-}32)$$

These results establish the Fourier transform pair

$$K\delta(t) \quad ⬭ \quad H(f) = K \qquad (2\text{-}33)$$

which is illustrated in Fig. 2-6.

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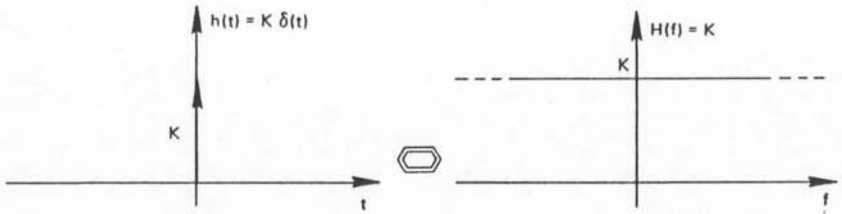

Figure 2-6. Fourier transform of an impulse function.

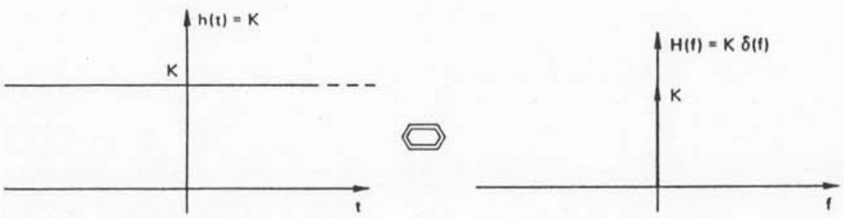

Figure 2-7. Fourier transform of a constant amplitude waveform.

Similarly the Fourier transform pair (Fig. 2-7)

$$h(t) = K \quad \Longleftrightarrow \quad K\delta(f) \qquad (2\text{-}34)$$

can be established where the reasoning process concerning existence is exactly as argued previously.

EXAMPLE 2-6

To illustrate the Fourier transform of periodic functions consider

$$h(t) = A\cos(2\pi f_0 t) \qquad (2\text{-}35)$$

The Fourier transform is given by

$$H(f) = \int_{-\infty}^{\infty} A\cos(2\pi f_0 t) e^{-j2\pi f t}\, dt$$

$$= \frac{A}{2}\int_{-\infty}^{\infty} [e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}] e^{-j2\pi f t}\, dt$$

$$= \frac{A}{2}\int_{-\infty}^{\infty} [e^{-j2\pi t(f - f_0)} + e^{-j2\pi t(f_0 + f)}]\, dt$$

$$= \frac{A}{2}\delta(f - f_0) + \frac{A}{2}\delta(f + f_0) \qquad (2\text{-}36)$$

where ar... nts identical to those leading to Eq. (2-32) have been employed. The inversion ...mula yields

$$h(t) = \int_{-\infty}^{\infty} \left[\frac{A}{2}\delta(f - f_0) + \frac{A}{2}\delta(f + f_0)\right] e^{j2\pi f t}\, df$$

$$= \frac{A}{2} e^{j2\pi f_0 t} + \frac{A}{2} e^{-j2\pi f_0 t}$$

$$= A\cos(2\pi f_0 t) \qquad (2\text{-}37)$$

Fourier transform pair

$$A\cos(2\pi f_0 t) \quad \Longleftrightarrow \quad \frac{A}{2}\delta(f - f_0) + \frac{A}{2}\delta(f + f_0) \qquad (2\text{-}38)$$

is illustrated in Fig. 2-8.

Similarly the Fourier transform pair (Fig. 2-9)

$$A\sin(2\pi f_0 t) \quad \Longleftrightarrow \quad j\frac{A}{2}\delta(f + f_0) - j\frac{A}{2}\delta(f - f_0) \qquad (2\text{-}39)$$

can be established. Note that the Fourier transform is imaginary.

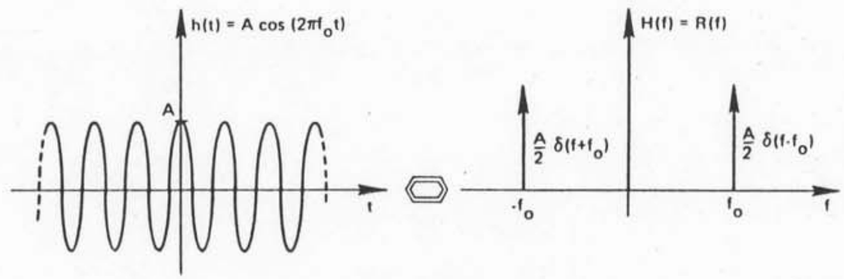

Figure 2-8. Fourier transform of $A\cos(at)$.

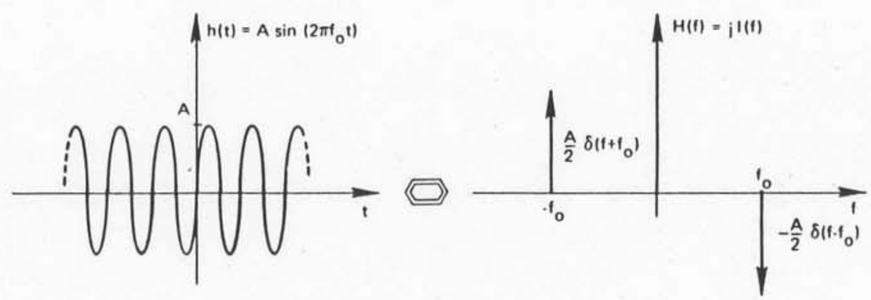

Figure 2-9. Fourier transform of $A\sin(at)$.

EXAMPLE 2-7

Without proof†, the Fourier transform of a sequence of equal distant impulse functions is another sequence of equal distant impulses;

†A Papoulis, *The Fourier Integral and Its Applications* (New York: McGraw-Hill, ... 44.

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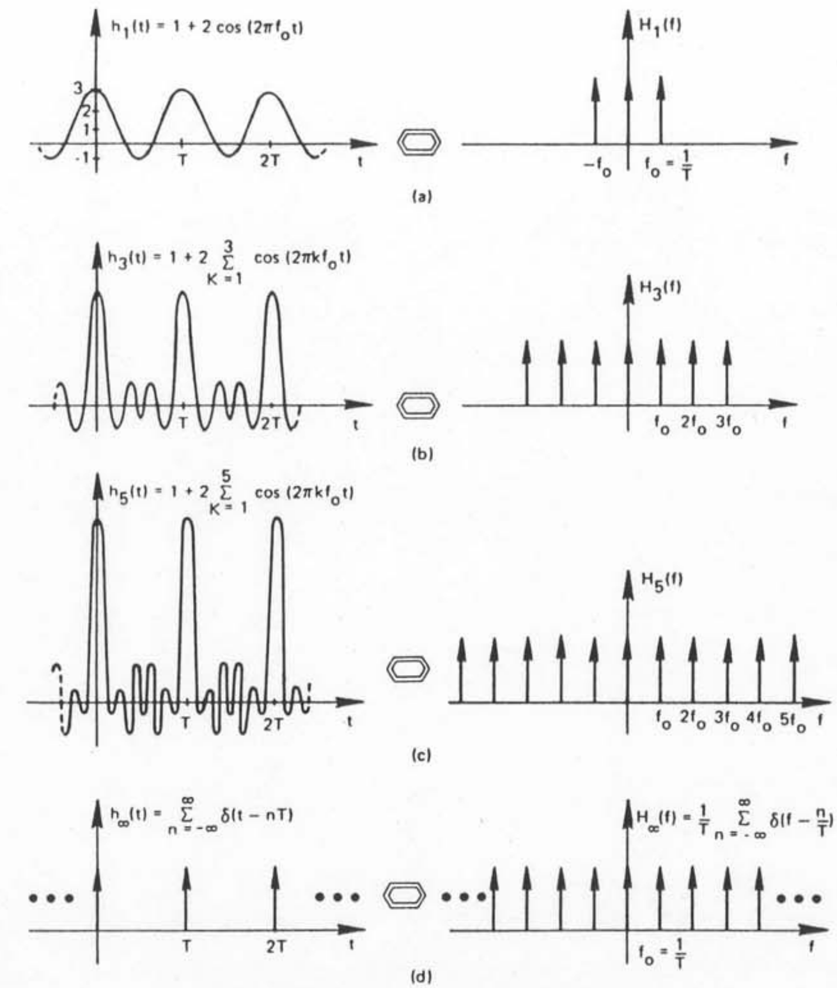

Figure 2-10. Graphical development of the Fourier transform of a sequence of equal distant impulse functions.

$$h(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad \Longleftrightarrow \quad H(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \tag{2-40}$$

A graphical development of this Fourier transform pair is illustrated in Fig. 2-10. The importance of Fourier transform pair (2-40) will become obvious in future discussions of discrete Fourier transforms.

Inversion Formula Proof

By means of distribution theory concepts it is possible to derive a simple formal proof of the inversion formula (2-5)

Substitution of $H(f)$ [Eq. (2-1)] into the inverse Fourier transform (2-5) yields

$$\int_{-\infty}^{\infty} H(f)e^{j2\pi ft}\,df = \int_{-\infty}^{\infty} e^{j2\pi ft}\,df \int_{-\infty}^{\infty} h(x)e^{-j2\pi fx}\,dx \tag{2-41}$$

Since [Eq. (A-21)]

$$\int_{-\infty}^{\infty} e^{j2\pi ft}\,dt = \delta(t)$$

then an interchange of integration in (2-41) gives

$$\int_{-\infty}^{\infty} H(f)e^{j2\pi ft}\,dt = \int_{-\infty}^{\infty} h(x)\,dx \int_{-\infty}^{\infty} e^{j2\pi f(t-x)}\,df$$
$$= \int_{-\infty}^{\infty} h(x)\,\delta(t - x)\,dx \tag{2-42}$$

But by the definition of the impulse function (2-28), Eq. (2-42) simply equals $h(x)$. This statement is valid only if $h(t)$ is continuous.† However if it is assumed that

$$h(t) = \frac{h(t^+) + h(t^-)}{2} \tag{2-43}$$

that is, if $h(t)$ is defined as the mid-value at a discontinuity, then the inversion formula still holds. Note that in the previous examples we carefully defined each discontinuous function consistent with Eq. (2-43).

2-4 ALTERNATE FOURIER TRANSFORM DEFINITIONS

It is a well established fact that the Fourier transform is a universally accepted tool of modern analysis. Yet to this day there is not a common definition of the Fourier integral and its inversion formula. To be specific the Fourier transform pair is defined as

$$H(\omega) = a_1 \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt \qquad \omega = 2\pi f \tag{2-44}$$

$$h(t) = a_2 \int_{-\infty}^{\infty} H(\omega)e^{j\omega t}\,d\omega \tag{2-45}$$

where the coefficients a_1 and a_2 assume different values depending on the user. Some set $a_1 = 1$; $a_2 = 1/2\pi$; others set $a_1 = a_2 = 1/\sqrt{2\pi}$, or set $a_1 = 1/2\pi$; $a_2 = 1$. Eqs. (2-44) and (2-45) impose the requirement that $a_1 a_2 = 1/2\pi$. Various users are then concerned with the splitting of the product $a_1 a_2$.

To resolve this question, we must define the relationship desired between the Fourier transform and the Laplace transform and the definition we wish to assume for the relationship between the total energy computed in the time

†See Appendix A. The definition of the impulse response is based on continuity of the function h(t).

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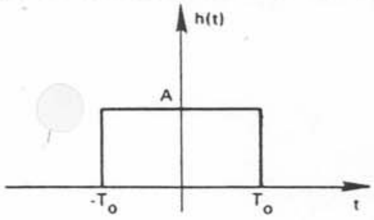
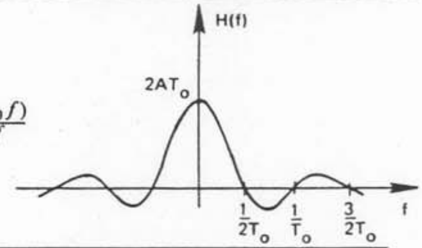
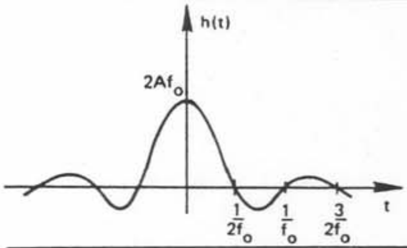
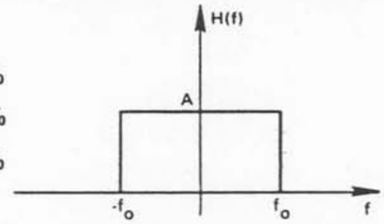
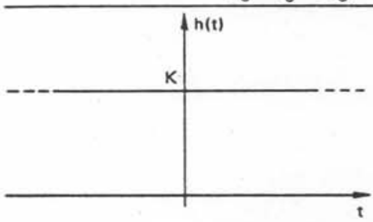
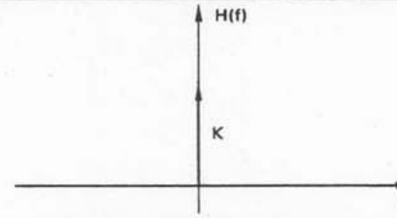
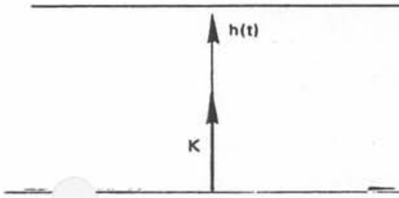
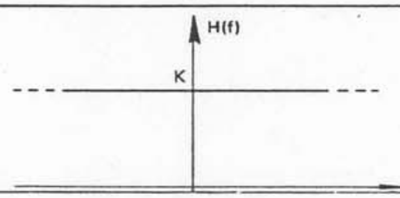
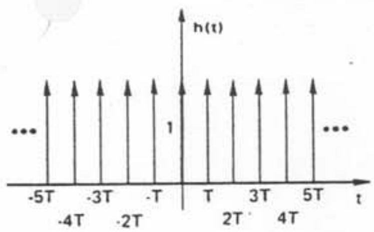
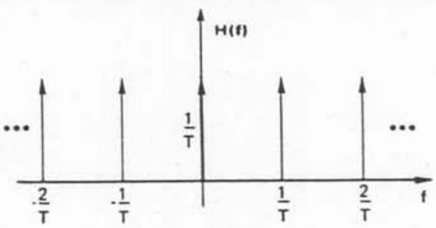
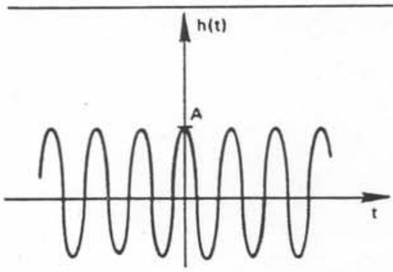
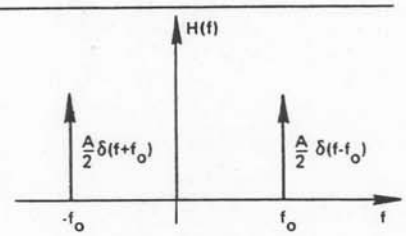
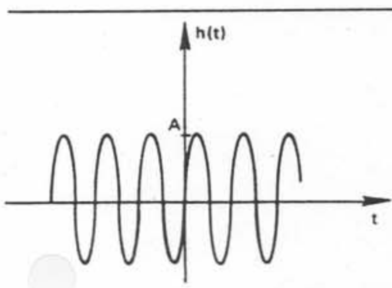
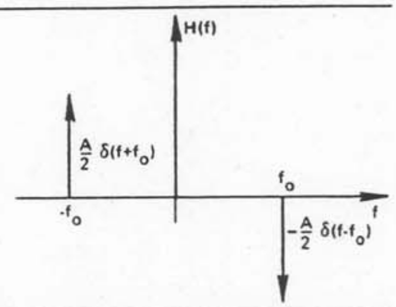
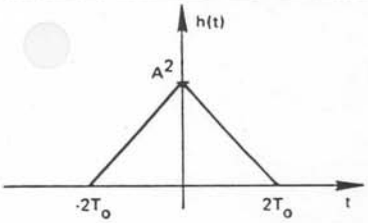
Time domain		Frequency domain
	$h(t)=A \quad \lvert t\rvert<T_0$; $=\frac{A}{2} \quad \lvert t\rvert=T_0$; $=0 \quad \lvert t\rvert>T_0$ ⟺ $H(f)=2AT_0\frac{\sin(2\pi T_0 f)}{2\pi T_0 f}$	
	$h(t)=2Af_0\frac{\sin(2\pi f_0 t)}{2\pi f_0 t}$ ⟺ $H(f)=A \quad \lvert f\rvert<f_0$; $=\frac{A}{2} \quad \lvert f\rvert=f_0$; $=0 \quad \lvert f\rvert>f_0$	
	$h(t)=K$ ⟺ $H(f)=K\delta(f)$	
	$h(t)=k\delta(t)$ ⟺ $H(f)=K$	
	$h(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$ ⟺ $H(f)=\frac{1}{T}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T}\right)$	
	$h(t)=A\cos(2\pi f_0 t)$ ⟺ $H(f)=\frac{A}{2}\delta(f-f_0)+\frac{A}{2}\delta(f+f_0)$	
	$h(t)=A\sin(2\pi f_0 t)$ ⟺ $H(f)=-j\frac{A}{2}\delta(f-f_0)+j\frac{A}{2}\delta(f+f_0)$	

Figure 2-11. Fourier transform pairs.

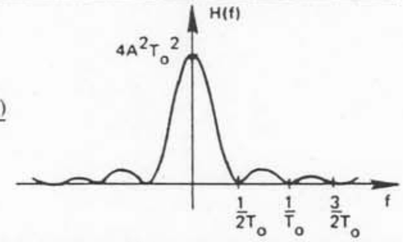
Time domain			Frequency domain

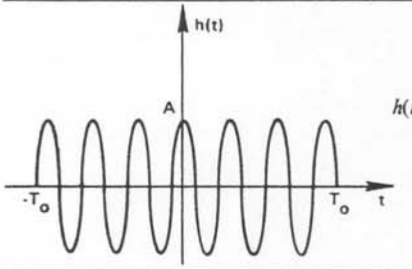


$$h(t) = -\frac{A^2}{2T_0}t + A^2 \quad |t| < 2T_0$$
$$= 0 \quad |t| > 2T_0$$

$\Longleftrightarrow$

$$H(f) = A^2\frac{\sin^2(2\pi T_0 f)}{(\pi f)^2}$$

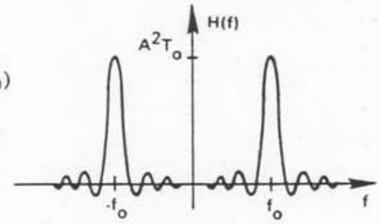


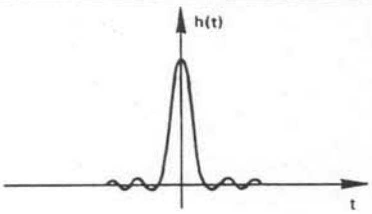


$$h(t) = A\cos(2\pi f_0 t) \quad |t| < T_0$$
$$= 0 \quad |t| > T_0$$

$\Longleftrightarrow$

$$H(f) = A^2 T_0[Q(f + f_0) + Q(f - f_0)]$$
$$Q(f) = \frac{\sin(2\pi T_0 f)}{2\pi T_0 f}$$

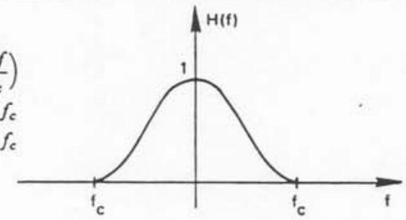


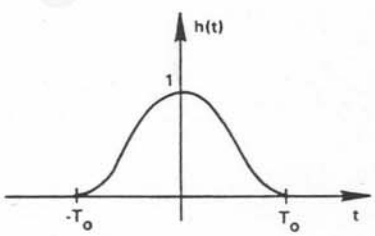


$$h(t) = \frac{1}{2}q(t) + \frac{1}{4}q\left(t + \frac{1}{2f_c}\right) + \frac{1}{4}q\left(t - \frac{1}{2f_c}\right)$$
$$q(t) = \frac{\sin(2\pi f_c t)}{\pi t}$$

$\Longleftrightarrow$

$$H(f) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi f}{f_c}\right) \quad |f| \leq f_c$$
$$= 0 \quad |f| > f_c$$

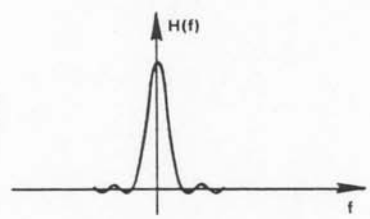


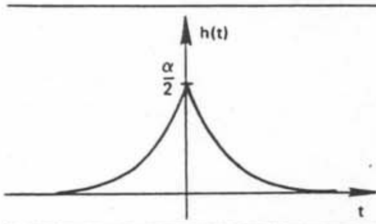


$$h(t) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi t}{T_0}\right) \quad |t| \leq T_0$$
$$= 0 \quad |t| > T_0$$

$\Longleftrightarrow$

$$H(f) = \frac{1}{2}Q(f) + \frac{1}{4}\left[Q\left(f + \frac{1}{2T_0}\right) + Q\left(f - \frac{1}{2T_0}\right)\right]$$
$$Q(f) = \frac{\sin(2\pi T_0 f)}{\pi f}$$

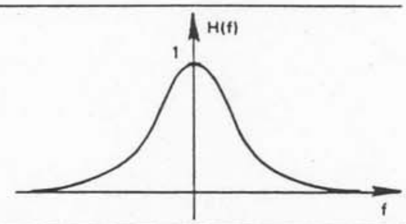


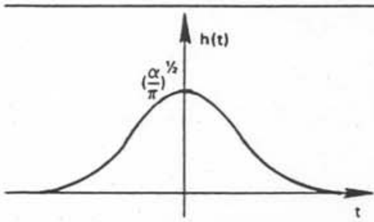


$$h(t) = \frac{1}{2}\alpha\exp(-\alpha|t|)$$

$\Longleftrightarrow$

$$H(f) = \frac{\alpha^2}{\alpha^2 + 4\pi^2 f^2}$$





$$h(t) = \left(\frac{\alpha}{\pi}\right)^{1/2}\exp(-\alpha t^2)$$

$\Longleftrightarrow$

$$H(f) = \exp\left(\frac{-\pi^2 f^2}{\alpha}\right)$$

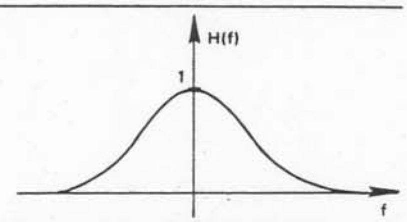


Figure 2-11 (continued).

domain and the total energy computed in ω, the radian frequency domain. For example, Parseval's Theorem (to be derived in Chapter 4) states:

$$\int_{-\infty}^{\infty} h^2(t)\,dt = 2\pi a_1^2 \int_{-\infty}^{\infty} |H(\omega)|^2\,d\omega \qquad (2\text{-}46)$$

If the energy computed in t is required to be equal to the energy computed in ω, then $a_1 = 1/\sqrt{2\pi}$. However, if the requirement is made that the Laplace transform, universally defined as

$$L[h(t)] = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt = \int_{-\infty}^{\infty} h(t)e^{-(\alpha+j\omega)t}\,dt \qquad (2\text{-}47)$$

shall reduce to the Fourier transform when the real part of s is set to zero, then a comparison of Eqs. (2-44) and (2-47) requires $a_2 = 1$, i.e., $a_1 = 1/2\pi$ which is in contradiction to the previous hypothesis.

A logical way to resolve this conflict is to define the Fourier transform pair as follows:

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}\,dt \qquad (2\text{-}48)$$

$$h(t) = \int_{-\infty}^{\infty} H(f)e^{j2\pi ft}\,df \qquad (2\text{-}49)$$

With this definition Parseval's Theorem becomes

$$\int_{-\infty}^{\infty} h^2(t)\,dt = \int_{-\infty}^{\infty} |H(f)|^2\,df$$

and Eq. (2-48) is consistent with the definition of the Laplace transform. Note that as long as integration is with respect to f, the scale factor $1/2\pi$ never appears. For this reason, the latter definition of the Fourier transform pair was chosen for this book.

2-5 FOURIER TRANSFORM PAIRS

A pictorial table of Fourier transform pairs is given in Fig. 2-11. This graphical and analytical dictionary is by no means complete but does contain the most frequently encountered transform pairs.

PROBLEMS

2-1. Determine the real and imaginary parts of the Fourier transform of each of the following functions:

a. $h(t) = e^{-a|t|} \qquad -\infty < t < \infty$

b. $h(t) = \begin{cases} k & t > 0 \\ \frac{k}{2} & t = 0 \\ 0 & t < 0 \end{cases}$

c. $h(t) = \begin{cases} -A & t < 0 \\ 0 & t = 0 \\ A & t > 0 \end{cases}$

d. $h(t) = \begin{cases} A\cos(2\pi f_0 t) & t > 0 \\ \frac{A}{2} & t = 0 \\ 0 & t < 0 \end{cases}$

e. $h(t) = \begin{cases} A & a < t < b;\ a, b > 0 \\ \frac{A}{2} & t = a;\ t = b \\ 0 & \text{elsewhere} \end{cases}$

f. $h(t) = \begin{cases} Ae^{-\alpha t}\sin(2\pi f_0 t) & t \geq 0 \\ 0 & t < 0 \end{cases}$

g. $h(t) = \frac{1}{2}\left[\delta(t+a) + \delta(t-a) + \delta\left(t + \frac{a}{2}\right) + \delta\left(t - \frac{a}{2}\right)\right]$

2-2. Determine the amplitude spectrum $|H(f)|$ and phase $\theta(f)$ of the Fourier transform of $h(t)$:

a. $h(t) = \frac{1}{|t|} \qquad -\infty < t < \infty$

b. $h(t) = e^{-\pi t^2} \qquad -\infty < t < \infty$

c. $h(t) = A\sin(2\pi f_0 t) \qquad 0 \leq t < \infty$

d. $h(t) = Ae^{-\alpha t}\cos(2\pi f_0 t) \qquad 0 \leq t < \infty$

2-3. Determine the inverse Fourier transform of each of the following:

a. $H(f) = \frac{\alpha^2}{\alpha^2 + (2\pi f)^2}$

b. $H(f) = \frac{\sin(2\pi fT)\cos(2\pi fT)}{(2\pi f)}$

c. $H(f) = \frac{1}{(j2\pi f + \alpha)^2}$

d. $H(f) = \frac{1}{(j2\pi f + \alpha)^3}$

e. $H(f) = \frac{\beta}{(\alpha + j2\pi f)^2 + \beta^2}$

f. $H(f) = (1 - f^2)^2 \qquad |f| < 1$
$\quad = 0 \qquad \text{otherwise}$

g. $H(f) = \frac{f^3}{f^4 + \alpha}$

h. $H(f) = \frac{f^2 + \alpha}{f^4 + 2\alpha}$

i. $H(f) = \frac{f}{(f^2 + \alpha)(f^2 + 4\alpha)}$

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2. Bracewell, R., *The Fourier Transform and Its Applications.* New York: McGraw-Hill, 1965.
3. Campbell, G. A., and R. M. Foster, *Fourier Integrals for Practical Applications.* New York: Van Nostrand Reinhold, 1948.
4. Erdilyi, A., *Tables of Integral Transforms,* Vol. 1. New York: McGraw-Hill, 1954.
5. Papoulis, A., *The Fourier Integral and Its Applications.* New York: McGraw-Hill, 1962.

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3

FOURIER TRANSFORM PROPERTIES

In dealing with Fourier transforms there are a few properties which are basic to a thorough understanding. A visual interpretation of these fundamental properties is of equal importance to knowledge of their mathematical relationships. The purpose of this chapter is to develop not only the theoretical concepts of the basic Fourier transform pairs, but also the *meaning* of these properties. For this reason we use ample analytical and graphical examples.

3-1 LINEARITY

If $x(t)$ and $y(t)$ have the Fourier transforms $X(f)$ and $Y(f)$, respectively, then the sum $x(t) + y(t)$ has the Fourier transform $X(f) + Y(f)$. This property is established as follows:

$$\int_{-\infty}^{\infty} [x(t) + y(t)]e^{-j2\pi ft}\,dt = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt + \int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}\,dt$$

$$= X(f) + Y(f) \tag{3-1}$$

Fourier transform pair

$$x(t) + y(t) \quad \Longleftrightarrow \quad X(f) + Y(f) \tag{3-2}$$

is of considerable importance because it reflects the applicability of the Fourier transform to linear system analysis.

Example 3-1

To illustrate the linearity property, consider the Fourier transform pairs

$$x(t) = K \quad \Longleftrightarrow \quad X(f) = K\delta(f) \tag{3-3}$$

$$y(t) = A\cos(2\pi f_0 t) \quad \Longleftrightarrow \quad Y(f) = \frac{A}{2}\delta(f - f_0) + \frac{A}{2}\delta(f + f_0) \tag{3-4}$$

By the linearity theorem

$$x(t) + y(t) = K + A\cos(2\pi f_0 t) \quad \Longleftrightarrow \quad X(f) + Y(f) = K\delta(f) + \frac{A}{2}\delta(f - f_0) + \frac{A}{2}\delta(f + f_0) \tag{3-5}$$

Figures 3-1(a), (b), and (c), illustrate each of the Fourier transform pairs, respectively.

3-2 SYMMETRY

If $h(t)$ and $H(f)$ are a Fourier transform pair then

$$H(t) \quad \Longleftrightarrow \quad h(-f) \tag{3-6}$$

Fourier transform pair (3-6) is established by rewriting Eq. (2-5)

$$h(-t) = \int_{-\infty}^{\infty} H(f)e^{-j2\pi ft}\,df \tag{3-7}$$

and by interchanging the parameters t and f

$$h(-f) = \int_{-\infty}^{\infty} H(t)e^{-j2\pi ft}\,dt \tag{3-8}$$

Example 3-2

To illustrate this property consider the Fourier transform pair

$$h(t) = A \qquad |t| < T_0 \quad \Longleftrightarrow \quad \frac{2AT_0 \sin(2\pi T_0 f)}{2\pi T_0 f} \tag{3-9}$$

illustrated previously in Fig. 2-3. By the symmetry theorem

$$2AT_0 \frac{\sin(2\pi T_0 t)}{2\pi T_0 t} \quad \Longleftrightarrow \quad h(-f) = h(f) = A \qquad |f| < T_0 \tag{3-10}$$

which is identical to the Fourier transform pair (2-27) illustrated in Fig. 2-5. Utilization of the symmetry theorem can eliminate many complicated mathematical developments; a case in point is the development of the Fourier transform pair (2-27).

3-3 TIME SCALING

If the Fourier transform of $h(t)$ is $H(f)$, then the Fourier transform of $h(kt)$ where k is a real constant greater than zero is determined by substituting $t' = kt$ in the Fourier integral equation;

$$\int_{-\infty}^{\infty} h(kt)e^{-j2\pi ft}\,dt = \int_{-\infty}^{\infty} h(t')e^{-j2\pi t'(f/k)}\frac{dt'}{k} = \frac{1}{k}H\left(\frac{f}{k}\right) \tag{3-11}$$

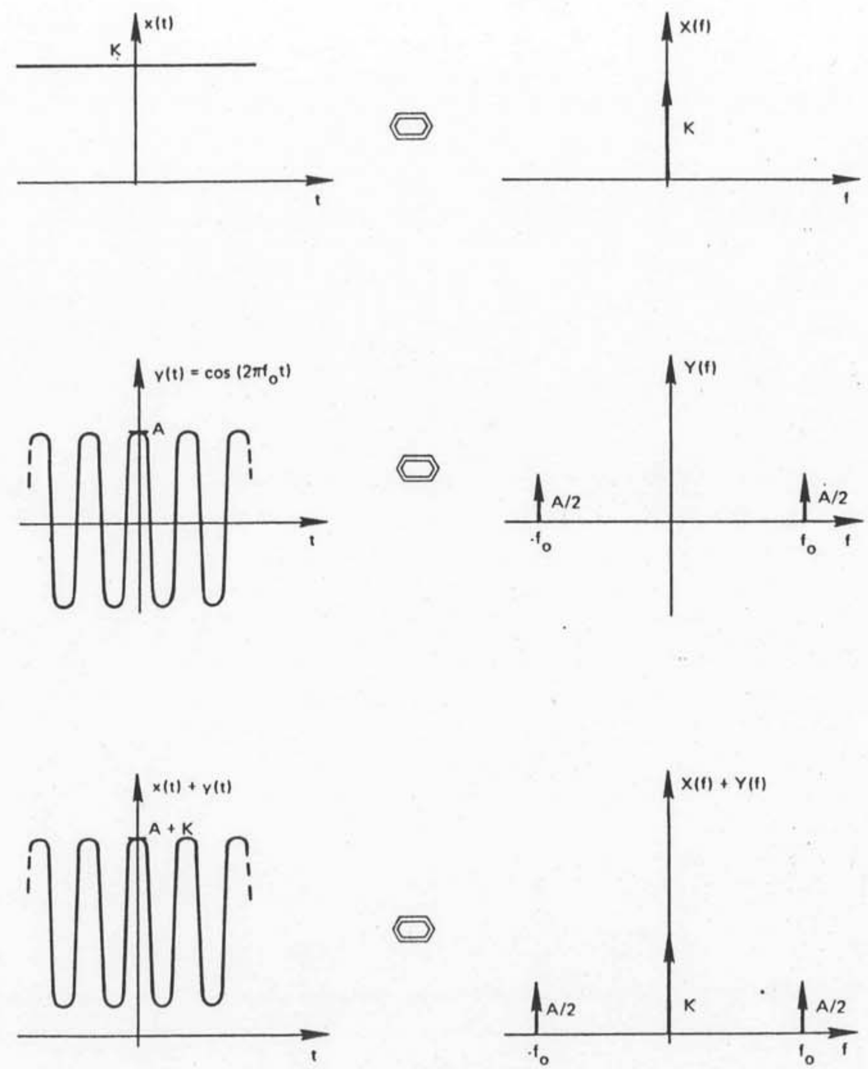

Figure 3-1. The linearity property.

For k negative, the term on the right-hand side changes sign because the limits of integration are interchanged. Therefore, time scaling results in the Fourier transform pair

$$h(kt) \quad \Longleftrightarrow \quad \frac{1}{|k|}H\left(\frac{f}{k}\right) \tag{3-12}$$

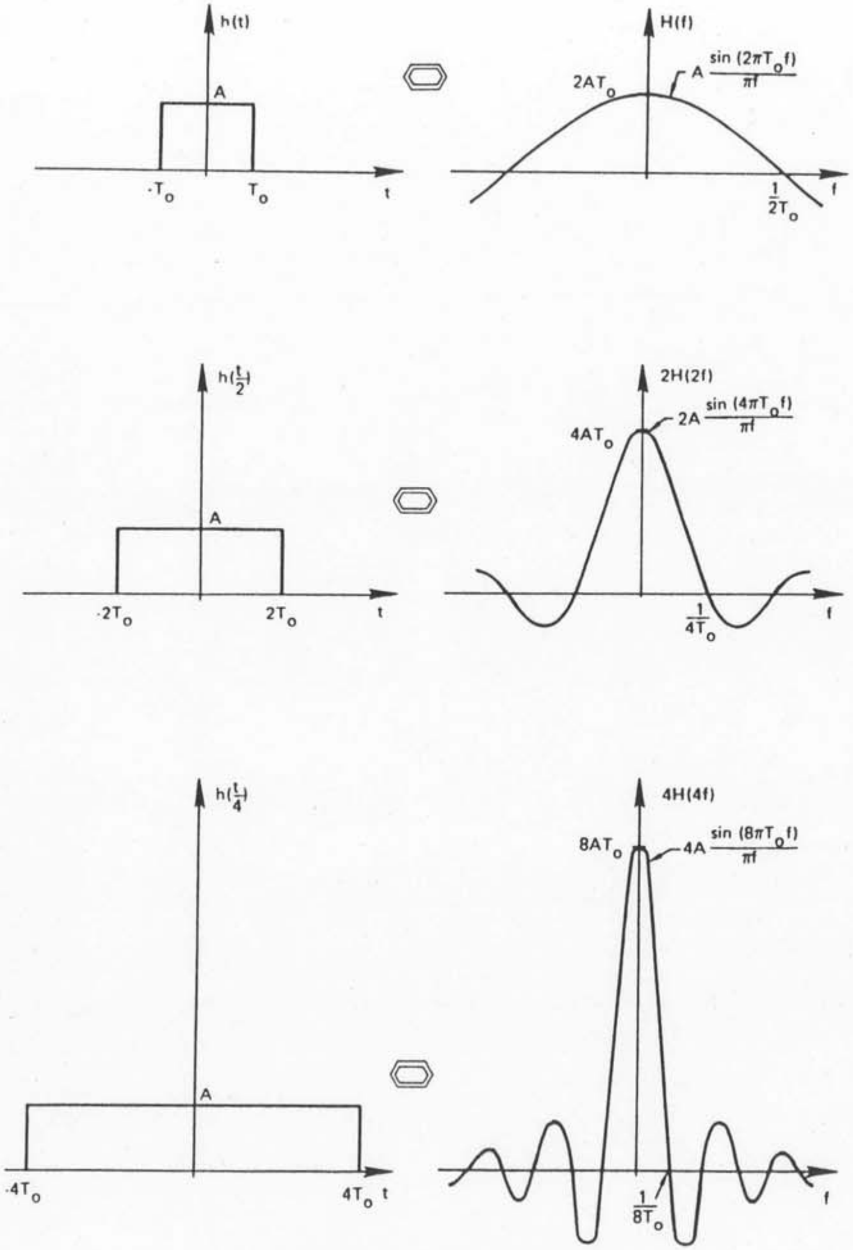

Figure 3-2. Time scaling property.

When dealing with time scaling of impulses, extra care must be exercised; from Eq. (A-10)

$$\delta(at) = \frac{1}{|a|}\delta(t) \tag{3-13}$$

EXAMPLE 3-3

The time scaling Fourier transform property is well-known in many fields of scientific endeavor. As shown in Fig. 3-2 time scale expansion corresponds to frequency scale compression. Note that as the time scale expands, the frequency scale not only contracts but the amplitude increases vertically in such a way as to keep the area constant. This is a well-known concept in radar and antenna theory.

3-4 FREQUENCY SCALING

If the inverse Fourier transform of $H(f)$ is $h(t)$, the inverse Fourier transform of $H(kf)$; k a real constant; is given by the Fourier transform pair

$$\frac{1}{|k|}h\left(\frac{t}{k}\right) \Leftrightarrow H(kf) \tag{3-14}$$

Relationship (3-14) is established by substituting $f' = kf$ into the inversion formula;

$$\int_{-\infty}^{\infty} H(kf)e^{j2\pi ft}\,df = \int_{-\infty}^{\infty} H(f')e^{j2\pi f'(t/k)}\frac{df'}{k} = \frac{1}{|k|}h\left(\frac{t}{k}\right) \tag{3-15}$$

Frequency scaling of impulse functions is given by

$$\delta(af) = \frac{1}{|a|}\delta(f) \tag{3-16}$$

EXAMPLE 3-4

Analogous to time scaling, frequency scale expansion results in a contraction of the time scale. This effect is illustrated in Fig. 3-3. Note that as the frequency scale expands, the amplitude of the time function increases. This is simply a reflection of the symmetry property (3-6) and the time scaling relationship (3-12).

EXAMPLE 3-5

Many texts state Fourier transform pairs in terms of the radian frequency ω. For example, Papoulis [2, page 44] gives

$$h(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \Leftrightarrow H(\omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2n\pi}{T}\right) \tag{3-17}$$

By the frequency scaling relationship (3-16) we know that

$$\frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta\left[2\pi\left(f - \frac{n}{T}\right)\right] = \frac{1}{T}\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \tag{3-18}$$

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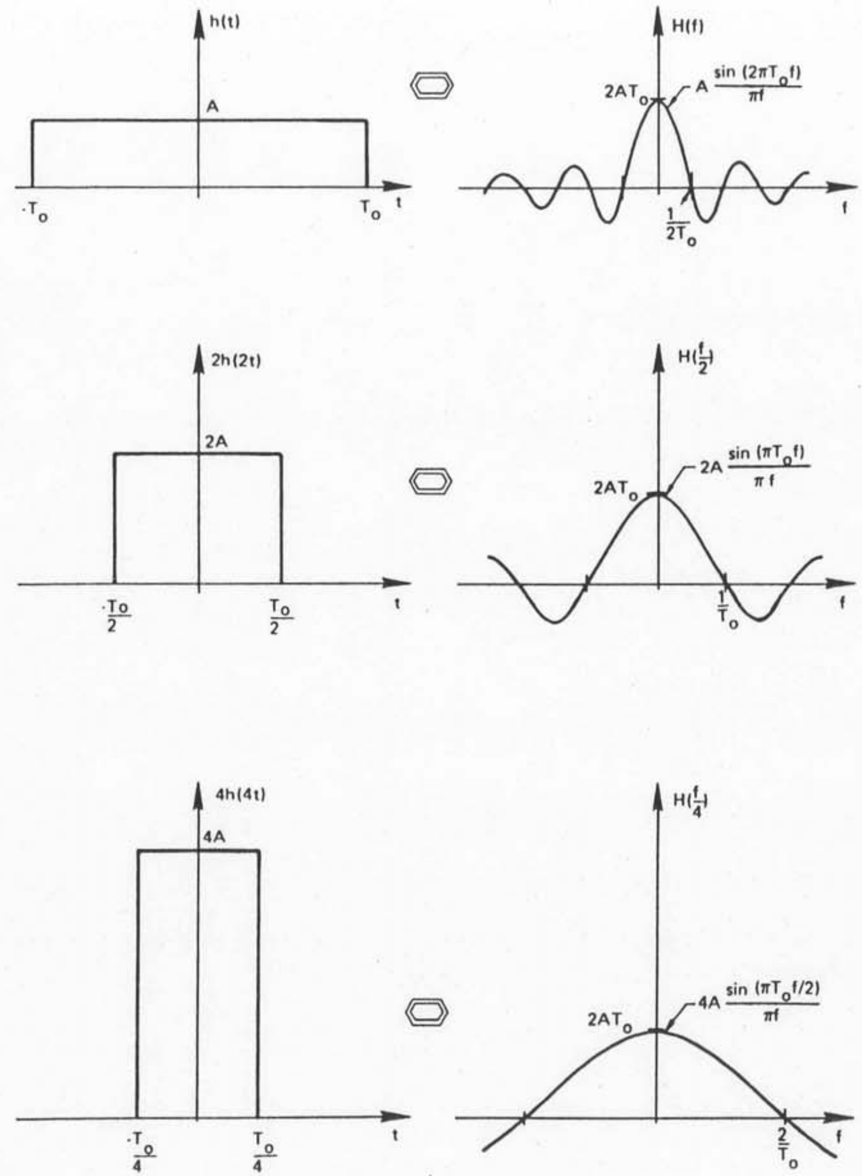

Figure 3-3. Frequency scaling property.

and (3-17) can be rewritten in terms of the frequency variable f

$$h(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad \Longleftrightarrow \quad H(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \tag{3-19}$$

which is Eq. (2-40).

3-5 TIME-SHIFTING

If $h(t)$ is shifted by a constant t_0 then by substituting $s = t - t_0$ the Fourier transform becomes

$$\begin{aligned}\int_{-\infty}^{\infty} h(t - t_0)e^{-j2\pi ft}\,dt &= \int_{-\infty}^{\infty} h(s)e^{-j2\pi f(s+t_0)}\,ds \\ &= e^{-j2\pi ft_0}\int_{-\infty}^{\infty} h(s)e^{-j2\pi fs}\,ds \\ &= e^{-j2\pi ft_0}H(f)\end{aligned} \tag{3-20}$$

The time-shifted Fourier transform pair is

$$h(t - t_0) \quad \Longleftrightarrow \quad H(f)e^{-j2\pi ft_0} \tag{3-21}$$

EXAMPLE 3-6

A pictorial description of this pair is illustrated in Fig. 3-4. As shown, time-shifting results in a change in the phase angle $\theta(f) = \tan^{-1}[I(f)/R(f)]$. Note that time-shifting does not alter the magnitude of the Fourier transform. This follows since

$$H(f)e^{-j2\pi ft_0} = H(f)[\cos(2\pi ft_0) - j\sin(2\pi ft_0)]$$

and hence the magnitude is given by

$$|H(f)e^{-j2\pi ft_0}| = \sqrt{H^2(f)[\cos^2(2\pi ft_0) + \sin^2(2\pi ft_0)]} = \sqrt{H^2(f)} \tag{3-22}$$

where $H(f)$ has been assumed to be real for simplicity. These results are easily extended to the case of $H(f)$, a complex function.

3-6 FREQUENCY SHIFTING

If $H(f)$ is shifted by a constant f_0, its inverse transform is multiplied by $e^{j2\pi tf_0}$

$$h(t)e^{j2\pi tf_0} \quad \Longleftrightarrow \quad H(f - f_0) \tag{3-23}$$

This Fourier transform pair is established by substituting $s = f - f_0$ into the inverse Fourier transform-defining relationship

$$\begin{aligned}\int_{-\infty}^{\infty} H(f - f_0)e^{j2\pi ft}\,dt &= \int_{-\infty}^{\infty} H(s)e^{j2\pi t(s+f_0)}\,ds \\ &= e^{j2\pi tf_0}\int_{-\infty}^{\infty} H(s)e^{j2\pi st}\,ds \\ &= e^{j2\pi tf_0}\,h(t)\end{aligned} \tag{3-24}$$

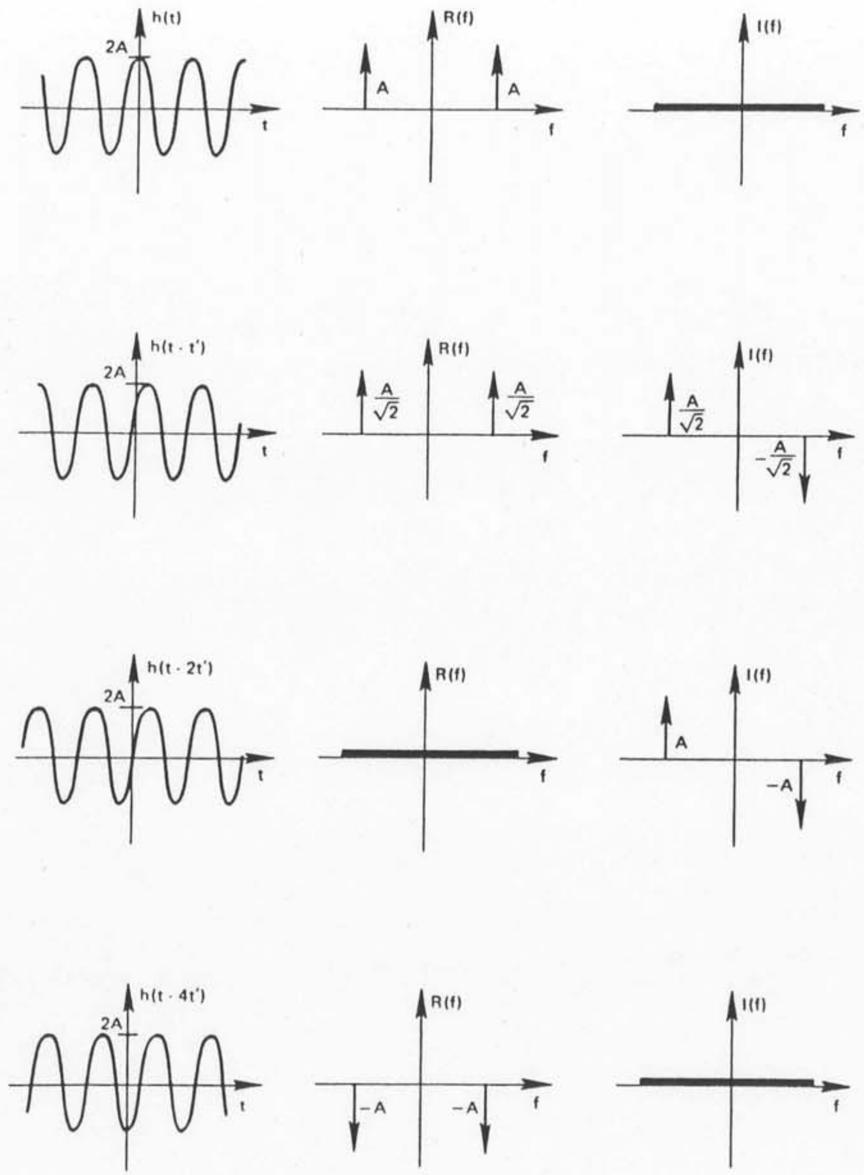

Figure 3-4. Time shifting property.

EXAMPLE 3-7

To illustrate the effect of frequency-shifting let us assume that the frequency function $H(f)$ is real. For this case, frequency-shifting results in a multiplication of the time function $h(t)$ by a cosine whose frequency is determined by the frequency shift f_0 (Fig. 3-5). This process is commonly known as modulation.

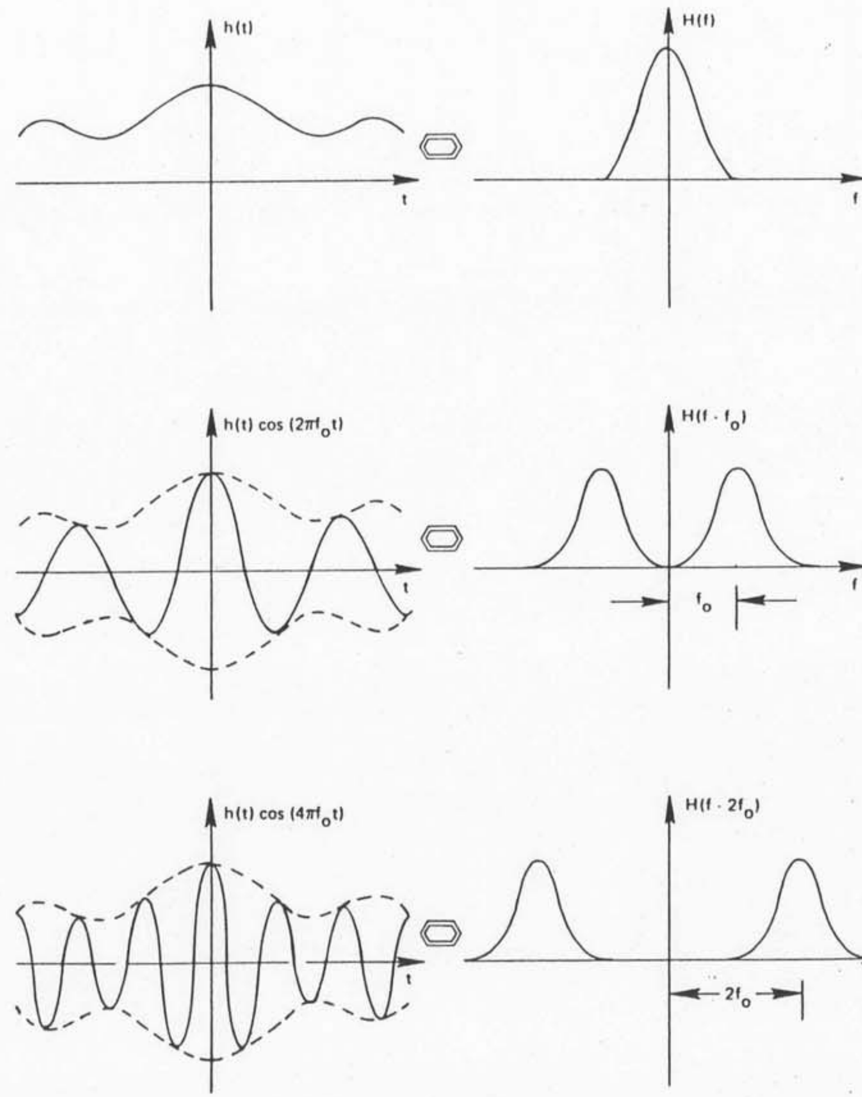

Figure 3-5. Frequency shifting property.

3-7 ALTERNATE INVERSION FORMULA

The inversion formula (2-5) may also be written as

$$h(t) = \left[\int_{-\infty}^{\infty} H^*(f)e^{-j2\pi ft}\,df\right]^* \tag{3-25}$$

where $H^*(f)$ is the conjugate of $H(f)$; that is, if $H(f) = R(f) + jI(f)$ then $H^*(f) = R(f) - jI(f)$. Relationship (3-25) is verified by simply performing the conjugation operations indicated.

$$\begin{aligned}
h(t) &= \left[\int_{-\infty}^{\infty} H^*(f)e^{-j2\pi ft}\,df\right]^* \\
&= \left[\int_{-\infty}^{\infty} R(f)e^{-j2\pi ft}\,df - j\int_{-\infty}^{\infty} I(f)e^{-j2\pi ft}\,df\right]^* \\
&= \left[\int_{-\infty}^{\infty} [R(f)\cos(2\pi ft) - I(f)\sin(2\pi ft)]\,df \right.\\
&\qquad \left. - j\int_{-\infty}^{\infty} [R(f)\sin(2\pi ft) + I(f)\cos(2\pi ft)]\,df\right]^* \\
&= \int_{-\infty}^{\infty} [R(f)\cos(2\pi ft) - I(f)\sin(2\pi ft)]\,df \\
&\qquad + j\int_{-\infty}^{\infty} [R(f)\sin(2\pi ft) + I(f)\cos(2\pi ft)]\,df \\
&= \int_{-\infty}^{\infty} [R(f) + jI(f)][\cos(2\pi ft) + j\sin(2\pi ft)]\,df \\
&= \int_{-\infty}^{\infty} H(f)e^{j2\pi ft}\,df
\end{aligned} \tag{3-26}$$

The significance of the alternate inversion formula is that now both the Fourier transform and its inverse contain the common term $e^{-j2\pi ft}$. This similarity will be of considerable importance in the development of fast Fourier transform computer programs.

3-8 EVEN FUNCTIONS

If $h_e(t)$ is an even function, that is, $h_e(t) = h_e(-t)$, then the Fourier transform of $h_e(t)$ is an even function and is real;

$$h_e(t) \Longleftrightarrow R_e(f) = \int_{-\infty}^{\infty} h_e(t)\cos(2\pi ft)\,dt \tag{3-27}$$

This pair is established by manipulating the defining relationships;

$$\begin{aligned}
H(f) &= \int_{-\infty}^{\infty} h_e(t)e^{-j2\pi ft}\,dt \\
&= \int_{-\infty}^{\infty} h_e(t)\cos(2\pi ft)\,dt - j\int_{-\infty}^{\infty} h_e(t)\sin(2\pi ft)\,dt
\end{aligned}$$

The imaginary term is zero since the integrand is an odd function. Since $\cos(2\pi ft)$ is an even function then $h_e(t)\cos(2\pi ft) = h_e(t)\cos[2\pi(-f)t]$ and $H_e(f) = H_e(-f)$; the frequency function is even. Similarly, if $H(f)$ is given as a real and even frequency function, the inversion formula yields

$$\begin{aligned}
h(t) &= \int_{-\infty}^{\infty} H_e(f)e^{j2\pi ft}\,dt = \int_{-\infty}^{\infty} R_e(f)e^{j2\pi ft}\,df \\
&= \int_{-\infty}^{\infty} R_e(f)\cos(2\pi ft)\,df + j\int_{-\infty}^{\infty} R_e(f)\sin(2\pi ft)\,df \\
&= \int_{-\infty}^{\infty} R_e(f)\cos(2\pi ft)\,df = h_e(t)
\end{aligned} \tag{3-29}$$

EXAMPLE 3-8

As shown in Fig. 3-6 the Fourier transform of an even time function is a real and even frequency function; conversely, the inverse Fourier transform of a real and even frequency function is an even function of time.

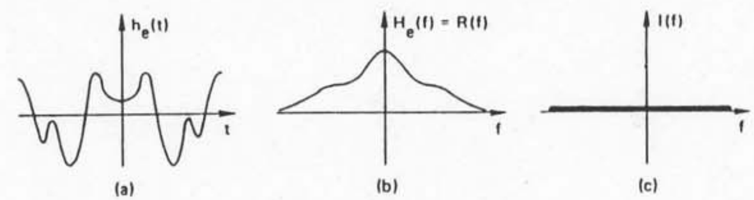

Figure 3-6. Fourier transform of an even function.

3-9 ODD FUNCTIONS

If $h_0(t) = -h_0(-t)$, then $h_0(t)$ is an odd function, and its Fourier transform is an odd and imaginary function,

$$\begin{aligned}
H(f) &= \int_{-\infty}^{\infty} h_0(t)e^{-j2\pi ft}\,dt \\
&= \int_{-\infty}^{\infty} h_0(t)\cos(2\pi ft)\,dt - j\int_{-\infty}^{\infty} h_0(t)\sin(2\pi ft)\,dt \\
&= -j\int_{-\infty}^{\infty} h_0(t)\sin(2\pi ft)\,dt = jI_0(f)
\end{aligned} \tag{3-30}$$

The real integral is zero since the multiplication of an odd and an even function is an odd function. Since $\sin(2\pi ft)$ is an odd function, then $h_0(t)\sin(2\pi ft) = -h_0(t)\sin[2\pi(-f)t]$ and $H_0(f) = -H_0(-f)$; the frequency function is odd. For $H(f)$ given as an odd and imaginary function, then

$$\begin{aligned}
h(t) &= \int_{-\infty}^{\infty} H(f)e^{j2\pi ft}\,dt = j\int_{-\infty}^{\infty} I_0(f)e^{j2\pi ft}\,df \\
&= j\int_{-\infty}^{\infty} I_0(f)\cos(2\pi ft)\,df + j\int_{-\infty}^{\infty} I_0(f)\sin(2\pi ft)\,df
\end{aligned}$$

and the resulting $h_0(t)$ is an odd function. The Fourier transform pair is thus established:

$$h_0(t) \quad \Longleftrightarrow \quad jI_0(f) = -j\int_{-\infty}^{\infty} h_0(t)\sin(2\pi f t)\,dt \tag{3-32}$$

EXAMPLE 3-9

An illustrative example of this transform pair is shown in Fig. 3-7. The function $h(t)$ depicted is odd; therefore, the Fourier transform is an odd and imaginary function of frequency. If a frequency function is odd and imaginary then its inverse transform is an odd function of time.

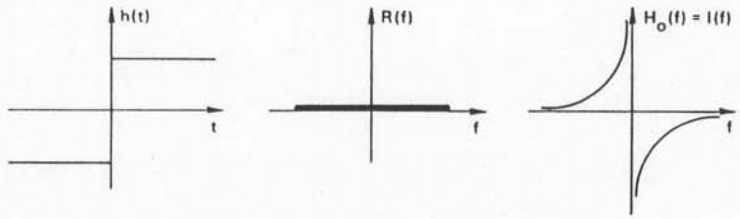

Figure 3-7. Fourier transform of an odd function.

3-10 WAVEFORM DECOMPOSITION

An arbitrary function can always be decomposed or separated into the sum of an even and an odd function;

$$\begin{aligned} h(t) &= \frac{h(t)}{2} + \frac{h(t)}{2} \\ &= \left[\frac{h(t)}{2} + \frac{h(-t)}{2}\right] + \left[\frac{h(t)}{2} - \frac{h(-t)}{2}\right] \\ &= h_e(t) + h_0(t) \end{aligned} \tag{3-33}$$

The terms in brackets satisfy the definition of an even and an odd function, respectively. From Eqs. (3-27) and (3-32) the Fourier transform of (3-33) is

$$H(f) = R(f) + jI(f) = H_e(f) + H_0(f) \tag{3-34}$$

where $H_e(f) = R(f)$ and $H_0(f) = jI(f)$. We will show in Chapter 10 that decomposition can be utilized to increase the speed of computation of the discrete Fourier transform.

EXAMPLE 3-10

To demonstrate the concept of waveform decomposition consider the exponential function [Fig. 3-8(a)]

$$h(t) = e^{-at} \qquad t \geq 0 \tag{3-35}$$

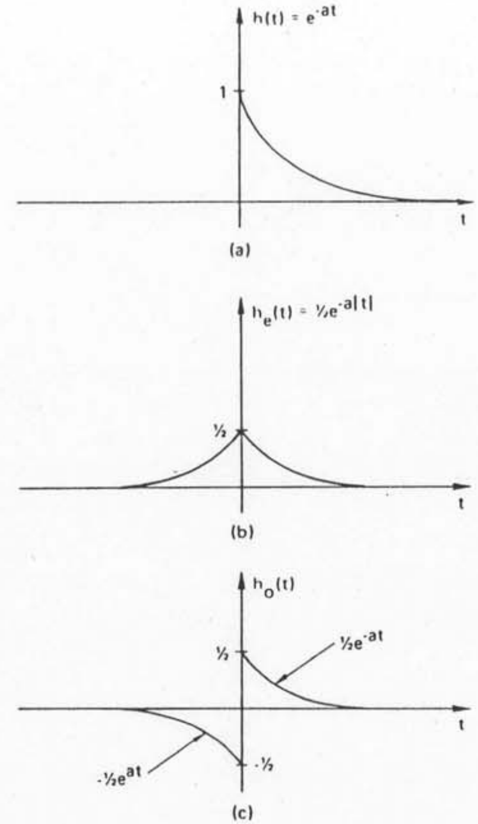

Figure 3-8. Waveform decomposition property.

Following the developments leading to (3-33) we obtain

$$\begin{aligned} h(t) &= \left[\frac{e^{-at}}{2}\right] + \left[\frac{e^{-at}}{2}\right] \\ &= \left\{\left[\frac{e^{-at}}{2}\right]_{t\geq 0} + \left[\frac{e^{at}}{2}\right]_{t\leq 0}\right\} + \left\{\left[\frac{e^{-at}}{2}\right]_{t\geq 0} - \left[\frac{e^{at}}{2}\right]_{t\leq 0}\right\} \\ &= \{e^{-a|t|}\} + \left\{\left[\frac{e^{-at}}{2}\right]_{t\geq 0} - \left[\frac{e^{at}}{2}\right]_{t\leq 0}\right\} \\ &= \{h_e(t)\} + \{h_0(t)\} \end{aligned} \tag{3-36}$$

Figures 3-8(b) and (c) illustrate the even and odd decomposition, respectively.

3-11 COMPLEX TIME FUNCTIONS

For ease of presentation we have to this point considered only real functions of time. The Fourier transform (2-1), the inversion integral (2-5), and

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and from (3-51) and (3-52)

$$H(f) = \left[\frac{R(f)}{2} + \frac{R(-f)}{2}\right] + j\left[\frac{I(f)}{2} - \frac{I(-f)}{2}\right] \tag{3-55}$$

$$G(f) = \left[\frac{I(f)}{2} + \frac{I(-f)}{2}\right] - j\left[\frac{R(f)}{2} - \frac{R(-f)}{2}\right] \tag{3-56}$$

Thus it is possible to separate the frequency function $Z(f)$ into the Fourier transforms of $h(t)$ and $g(t)$, respectively. As will be demonstrated in Chapter 10, this technique can be used advantageously to increase the speed of computation of the discrete Fourier transform.

3-12 SUMMARY OF PROPERTIES

For future reference the basic properties of the Fourier transform are summarized in Table 3-2. These relationships will be of considerable importance throughout the remainder of this book.

TABLE 3-2 PROPERTIES OF FOURIER TRANSFORMS

Time domain	Equation no.	Frequency domain
Linear addition $x(t) + y(t)$	(3-2)	Linear addition $X(f) + Y(f)$
Symmetry $H(t)$	(3-6)	Symmetry $h(-f)$
Time scaling $h(kt)$	(3-12)	Inverse scale change $\frac{1}{k}H\left(\frac{f}{k}\right)$
Inverse scale change $\frac{1}{k}h\left(\frac{t}{k}\right)$	(3-14)	Frequency scaling $H(kf)$
Time shifting $h(t - t_0)$	(3-21)	Phase shift $H(f)e^{-j2\pi f t_0}$
Modulation $h(t)e^{j2\pi t f_0}$	(3-23)	Frequency shifting $H(f - f_0)$
Even function $h_e(t)$	(3-27)	Real function $H_e(f) = R_e(f)$
Odd function $h_0(t)$	(3-30)	Imaginary $H_0(f) = jI_0(f)$
Real function $h(t) = h_r(t)$	(3-43) (3-44)	Real part even Imaginary part odd $H(f) = R_e(f) + jI_0(f)$
Imaginary function $h(t) = jh_i(t)$	(3-45) (3-46)	Real part odd Imaginary part even $H(f) = R_0(f) + jI_e(f)$

PROBLEMS

3-1. Let

$$h(t) = \begin{cases} A & |t| < 2 \\ \frac{A}{2} & t = \pm 2 \\ 0 & |t| > 2 \end{cases}$$

$$x(t) = \begin{cases} -A & |t| < 1 \\ -\frac{A}{2} & t = \pm 1 \\ 0 & |t| > 1 \end{cases}$$

Sketch $h(t)$, $x(t)$, and $[h(t) - x(t)]$. Use Fourier transform pair (2-21) and the linearity theorem to find the Fourier transform of $[h(t) - x(t)]$.

3-2. Consider the functions $h(t)$ illustrated in Fig. 3-9. Use the linearity property to derive the Fourier transform of $h(t)$.

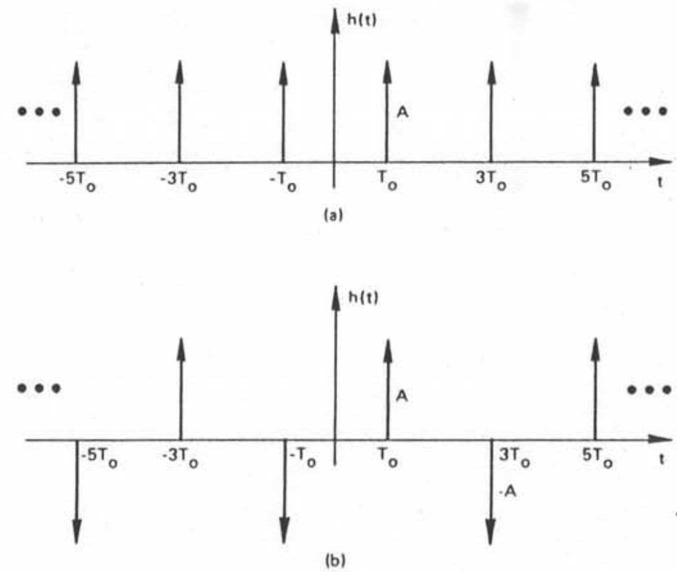

Figure 3-9.

3-3. Use the symmetry theorem and the Fourier transform pairs of Fig. 2-11 to determine the Fourier transform of the following:

a. $h(t) = \dfrac{A^2 \sin^2 (2\pi T_0 t)}{(\pi t)^2}$

b. $h(t) = \dfrac{\alpha^2}{(\alpha^2 + 4\pi^2 t^2)}$

c. $h(t) = \exp\left(\dfrac{-\pi^2 t^2}{\alpha}\right)$

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3-4. Derive the frequency scaling property from the time scaling property by means of the symmetry theorem.

3-5. Consider

$$h(t) = \begin{cases} A^2 - \dfrac{A^2|t|}{2T_0} & |t| < 2T_0 \\ 0 & |t| > 2T_0 \end{cases}$$

Sketch the Fourier transform of $h(2t)$, $h(4t)$, and $h(8t)$. (The Fourier transform of $h(t)$ is given in Fig. 2-11.)

3-6. Derive the time scaling property for the case k negative.

3-7. By means of the shifting theorem find the Fourier transform of the following functions:

a. $h(t) = \dfrac{A \sin [2\pi f_0(t - t_0)]}{\pi(t - t_0)}$

b. $h(t) = K\delta(t - t_0)$

c. $h(t) = \begin{cases} A^2 - \dfrac{A^2}{2T_0}|t - t_0| & |t - t_0| < 2T_0 \\ 0 & |t - t_0| > 2T_0 \end{cases}$

3-8. Show that

$$h(\alpha t - \beta) \Leftrightarrow \frac{1}{|\alpha|} e^{j2\pi\beta f/\alpha} H\left(\frac{f}{\alpha}\right)$$

3-9. Show that $|H(f)| = |e^{-j2\pi f t_0} H(f)|$; that is, the magnitude of a frequency function is independent of the time delay.

3-10. Find the inverse Fourier transform of the following functions by using the frequency shifting theorem:

a. $H(f) = \dfrac{A \sin [2\pi T_0(f - f_0)]}{\pi(f - f_0)}$

b. $H(f) = \dfrac{\alpha^2}{[\alpha^2 + 4\pi^2(f + f_0)^2]}$

c. $H(f) = \dfrac{A^2 \sin^2 [2\pi T_0(f - f_0)]}{[\pi(f - f_0)]^2}$

3-11. Review the derivations leading to Eqs. (2-9), (2-13), (2-20), (2-25), (2-26), and (2-32). Note the mathematics which result are real for the Fourier transform of an even function.

3-12. Decompose and sketch the even and odd components of the following functions:

a. $h(t) = \begin{cases} 1 & 1 < t < 2 \\ 0 & \text{otherwise} \end{cases}$

b. $h(t) = \dfrac{1}{[2 - (t - 2)^2]}$

c. $h(t) = \begin{cases} -t + 1 & 0 < t \leq 1 \\ 0 & \text{otherwise} \end{cases}$

3-13. Prove each of the properties listed in Table 3-1.

3-1[illegible] [illegible] $h(t)$ is real, show that $|H(f)|$ is an even function.

3-15. By making a substitution of variable in Eq. (2-28) show that

$$\int_{-\infty}^{\infty} x(t)\delta(at - t_0)\, dt = \frac{1}{a} x\left(\frac{t_0}{a}\right)$$

3-16. Prove the following Fourier transform pairs:

a. $\dfrac{dh(t)}{dt} \Leftrightarrow j2\pi f H(f)$

b. $[-j2\pi t]s(t) \Leftrightarrow \dfrac{dH(f)}{df}$

3-17. Use the derivative relationship of Problem 3-16(a) to find the Fourier transform of a pulse waveform given the Fourier transform of a triangular waveform.

REFERENCES

1. BRACEWELL, R., *The Fourier Transform and Its Applications.* New York: McGraw-Hill, 1965.

2. PAPOULIS, A., *The Fourier Integral and Its Applications.* New York: McGraw-Hill, 1962.

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CONVOLUTION & CORRELATION

Primary reference is the book *The Fast Fourier Transform* by E. O. Brigham. However, the chapters on convolution and correlation are unnecessarily too long in that book. The notes should be sufficient.

Convolution

Convolution is defined as

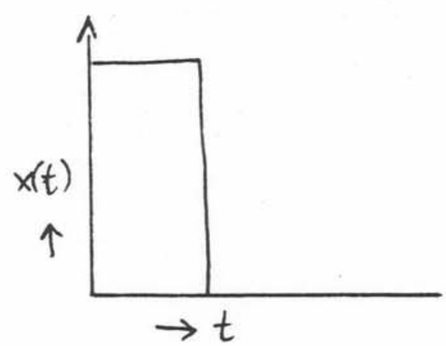
$$y(t) \equiv \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau$$

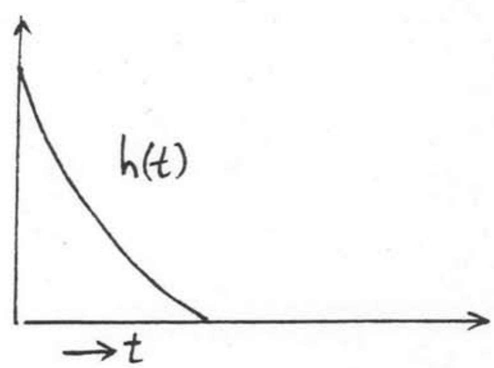
and is usually represented as

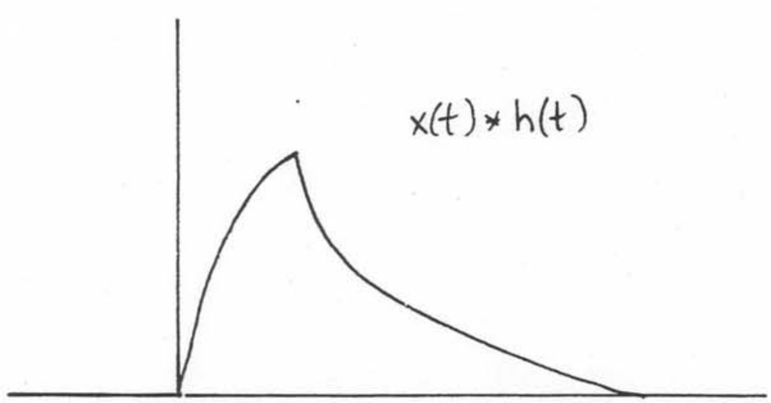
$$y(t) = x(t) * h(t).$$

Convolution consists of three steps:

1. Folding, $h(\tau) \rightarrow h(-\tau)$.
2. Displacement, $h(-\tau) \rightarrow h(t-\tau)$.
3. Multiplication, $h(t-\tau)x(\tau)$.
4. Integration, $\int h(t-\tau)x(\tau)d\tau$.







Note that a function convolved with a δ-function remains unchanged,

$$\begin{aligned} y(t) &= x(t) * \delta(t) \\ &= \int x(t)\delta(t-\tau)d\tau \\ &= x(t). \end{aligned}$$

Properties of Convolution:

Symmetric	$x(t) * h(t) = h(t) * x(t)$
Associative	$h(t) * [g(t) * x(t)] = [h(t) * g(t)] * x(t)$
Distributive	$h(t) * [g(t) + x(t)] = h(t) * g(t) + h(t) * x(t)$

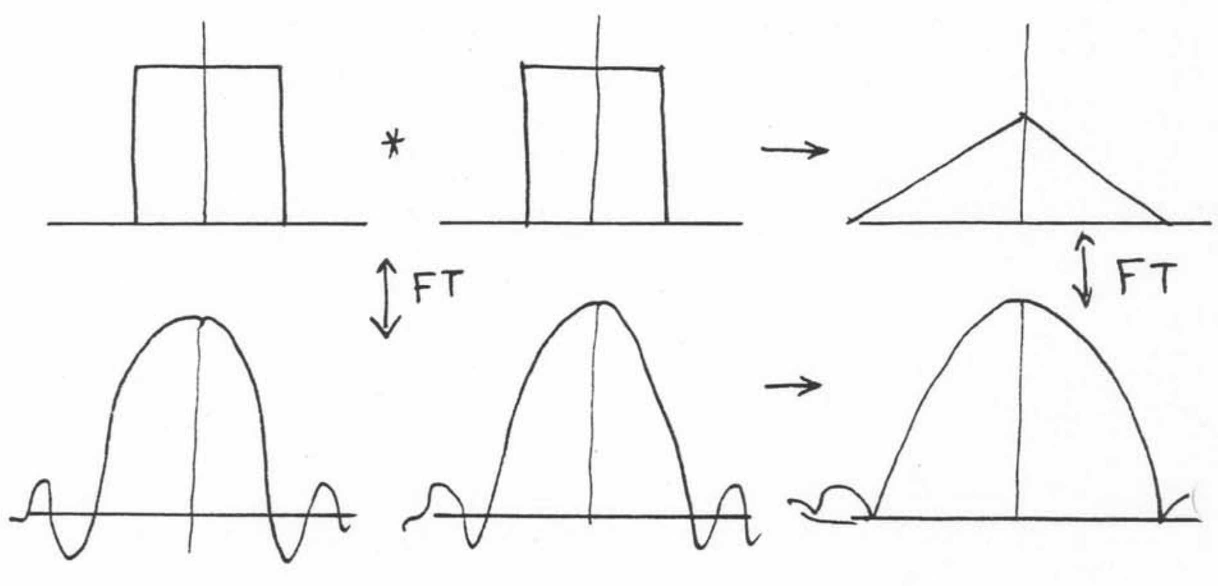
The Convolution Theorem.

$$\mathcal{F}[x(t) * h(t)] = \mathcal{F}[x(t)]\mathcal{F}[h(t)].$$

Proof:

$$\begin{aligned} \int_{-\infty}^{+\infty} y(t)e^{-2\pi jft}dt &= \int_{-\infty}^{+\infty} \Big[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\Big] e^{-2\pi jft}dt \\ &= \int_{-\infty}^{+\infty} x(\tau)\Big[\int_{-\infty}^{+\infty} h(t-\tau)e^{-2\pi jft}dt\Big] d\tau \\ &= \int_{-\infty}^{+\infty} x(\tau)e^{-2\pi jf\tau}H(f)d\tau \\ &= H(f)X(f) \end{aligned}$$

Example.



Parseval's Theorem. Consider $y(t) = h^2(t)$. Then

$$\mathcal{F}[h(t) \times h(t)] = \mathcal{F}[h(t)] * \mathcal{F}[h(t)]$$

or

$$\int_{-\infty}^{+\infty} h^2(t) e^{-2\pi j \sigma t} dt = \int_{-\infty}^{+\infty} H(f) H(\sigma - f) df.$$

Let $\sigma = 0$ then we have

$$\int_{-\infty}^{+\infty} h^2(t) dt = \int_{-\infty}^{+\infty} |H(f)|^2 df.$$ if signal is real $H(-f) = H^*(f)$

Thus the energy integrated in frequency space is equal to that in the time domain.

Cross- & Auto-Correlation

Cross-correlation of $x(t)$ with $h(t)$ is defined to be

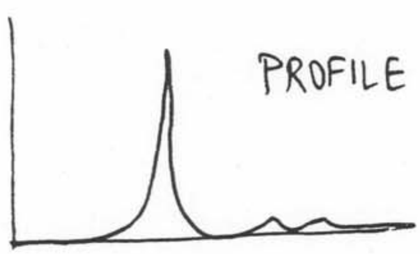
$$z(t) \equiv \int_{-\infty}^{+\infty} x(\tau) h(t + \tau) d\tau.$$

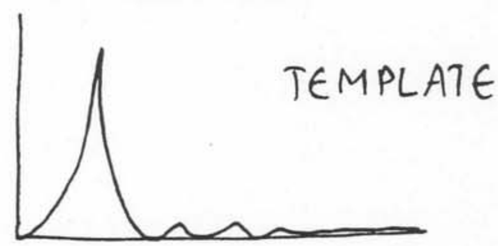
The rules of algebra are the same as for convolution. Cross-correlation is used to extract weak signals from noise. The basic idea is to correlate the weak signal buried in noise (represented by say $x(t)$). It is easy to see that $z(t)$ will peak up when $h(t + \tau)$ is similar to the weak signal that is buried in $x(t)$.

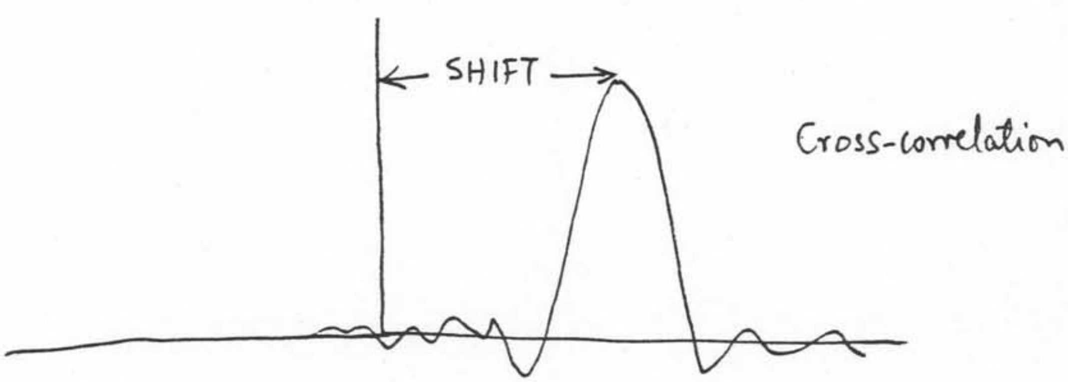
Just like the convolution theorem we have a correlation theorem.

$$\mathcal{F} \int_{-\infty}^{+\infty} h(\tau) x(t + \tau) dt = H(f) X^*(f).$$

The quantity on the right side is called as the cross-power spectrum between $H(f)$ and $X(f)$. This relation plays a central role in radio interferometers. Two common applications of cross-correlation are in determination of red shift of galaxies and the arrival times of pulsar signals.







Auto-correlation

This is a particular case of cross-correlation,

$$C(\tau) = \int_{-\infty}^{+\infty} x(t)x(t+\tau)d\tau.$$

It is easy to demonstrate that $C(\tau)$ is symmetrical and that

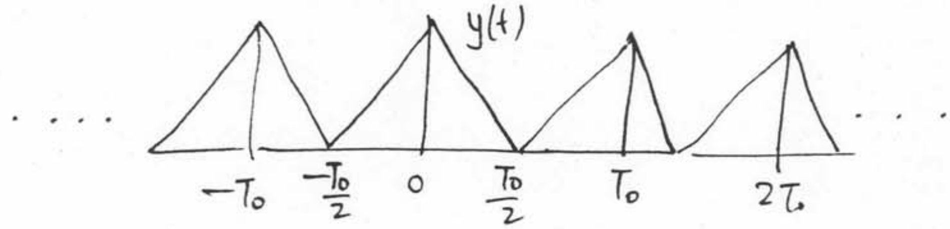
$$\mathcal{F}[C(\tau)] = |X(f)|^2.$$

The quantity on the right side is referred to as the Power Spectrum. This relation is the basis of astronomical spectroscopy at cm and mm wavelengths.

Note: For the special case where either $x(t)$ or $h(t)$ are symmetrical the cross-correlation and convolution are the same.

FOURIER SERIES & Sampled Waveforms

The usual approach to Fourier series is to consider the Fourier series as a special case of periodic functions



The Fourier coefficients are

$$\alpha_k = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{T_0/2} y(t)\, e^{-j 2\pi k f_0 t}\, dt$$

$$f_0 = 1/T_0$$

$$h = 0, \pm 1, \pm 2 \dots$$

The physical basis of the above formula is clear: a periodic waveform in T_0 can only have frequencies, k

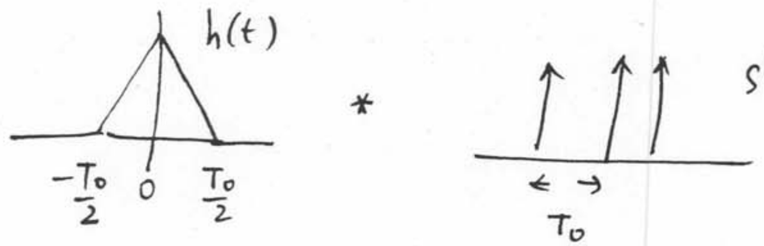
Fourier series and sampled wave-forms can be shown to be special cases of continuous Fourier transform. Indeed, sampled wave-forms lead naturally to the discrete Fourier transform.

Fourier series: convolution of $h(t)$ with $sha(T_0)$ $(= \Sigma\, \delta(t - nT_0)$

Sampled waveform: multiplication of $h(t)$ with $sha(T)$.

FOURIER SERIES:

We can regard $y(t) = h(t) * sha(T_0)$



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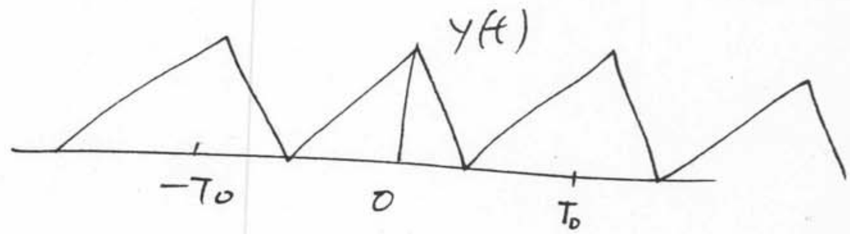


$$\mathcal{F}(y) = \mathcal{F}(h * sha)$$

$$Y(f) = H(f) \cdot \frac{1}{T_0} \sum_{-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right)$$

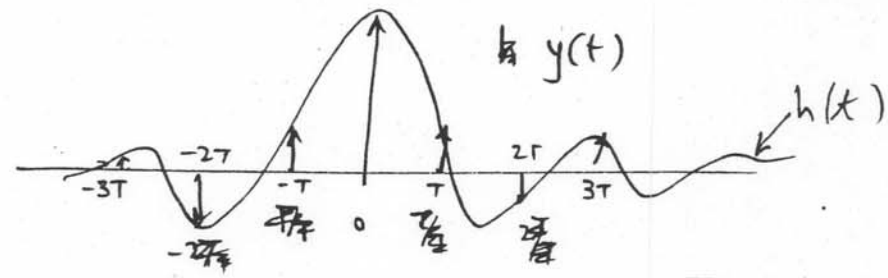
$$Y(f) = \frac{1}{T_0} \sum_{-\infty}^{\infty} H\left(\frac{n}{f_0}\right) \delta\left(f - \frac{n}{T_0}\right)$$

↑ note scaling factor

This expression immediately justifies the physical basis for the usual formula for Fourier Series.

SAMPLED SERIES:

Consider a wave-form $h_s(t)$ that is sampled at the rate $f_s = 1/T$.



We can regard $y(t) = h(t) \cdot \sum \delta(t - nT)$ where $h(t)$ is the continuous waveform and $y(t)$ is the sampled series.

$$y(t) = \sum h(nT)\, \delta(t - nT)$$

$$= h(t) \cdot \sum_{-\infty}^{\infty} \delta(t - nT)$$

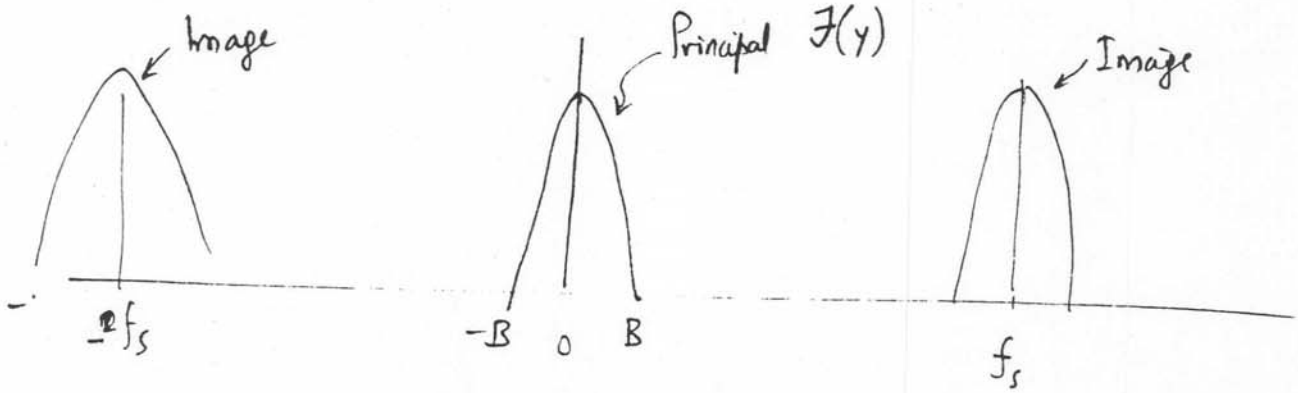
$$\therefore \mathcal{F}(y) = \mathcal{F}(h) * \mathcal{F}\left(\sum_{-\infty}^{\infty} \delta(t - nT)\right)$$

$$Y(f) = H(f) * \Delta(f_s)$$

Fourier transform of sampled series ($Y(f)$)

Fourier transform of continuous waveform ($H(f)$)

$\Delta(f_s)$: $\frac{1}{T} \sum \delta(f - nf_s)$, $f_s = 1/T$



Thus the process of sampling imposes periodicity in the data. This leads to periodicity in frequency.

In particular, we have more frequency structure than before. This is not surprising. A δ-function has infinite frequency structure and hence $h(t) . \delta(t - nT)$ has more frequency content than $h(t)$ alone

You will immediately note that as long as

$$f_s > 2B \qquad B = \text{"bandwidth"}$$

then the images do not overlap.

This is the famous sampling theorem.

I prefer to think 2B as the total frequency content of the signal (positive & negative frequen[cies])

$$\therefore \boxed{\text{Sampling rate} \geqslant \text{Frequency content}}$$

The Sampling Theorem: An interpolation formula

Consider a sampled time series

$$\hat{h}(t) = h(nT) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

where T is the sample-sample separation.

Then the sampling theorem states that $h(t)$, the analog signal is given by

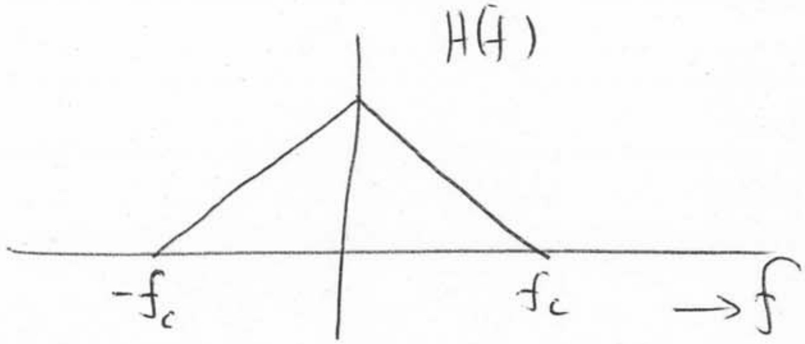
$$h(t) = T \sum_{n=-\infty}^{\infty} h(nT) \sin \frac{2\pi f_c (t - nT)}{\pi (t - nT)}$$

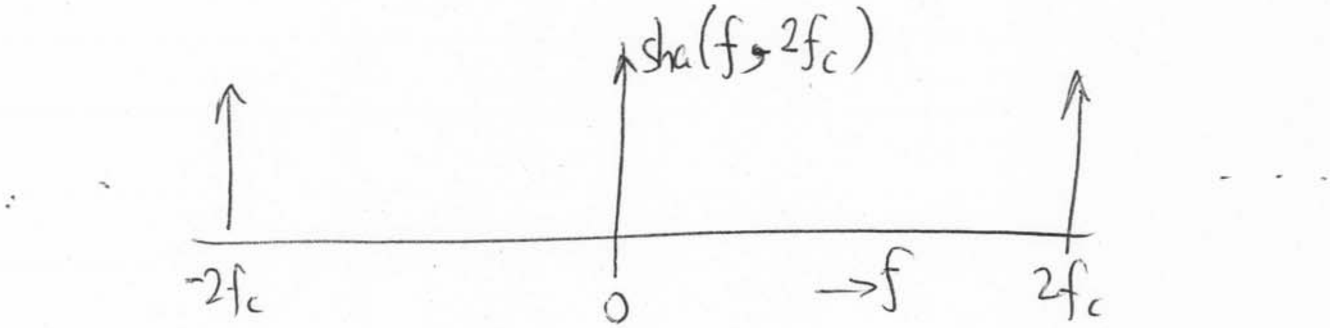
provided $T = \frac{1}{2f_c}$ and bandwidth of the signal is $B \leq f_c$

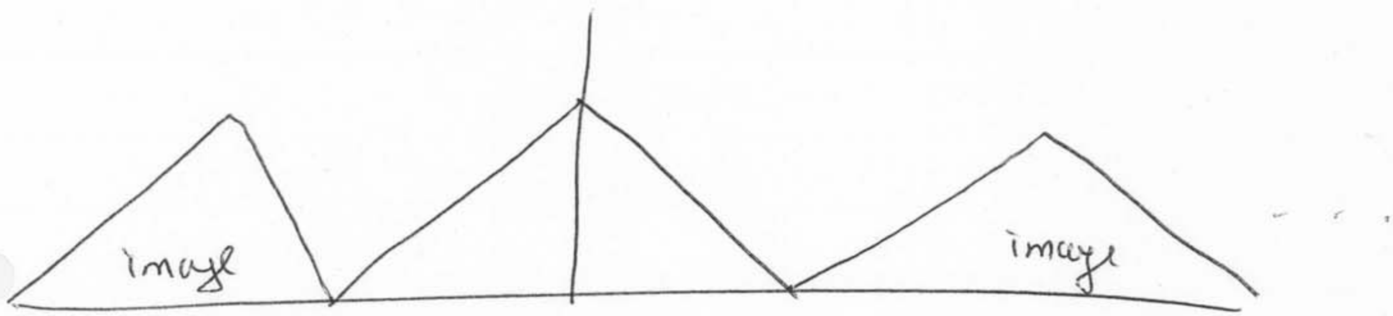
<u>Proof:</u>

$$\mathcal{F}(\hat{h}) = H(f) * \frac{1}{T} \text{sha}(f - \tfrac{1}{T})$$

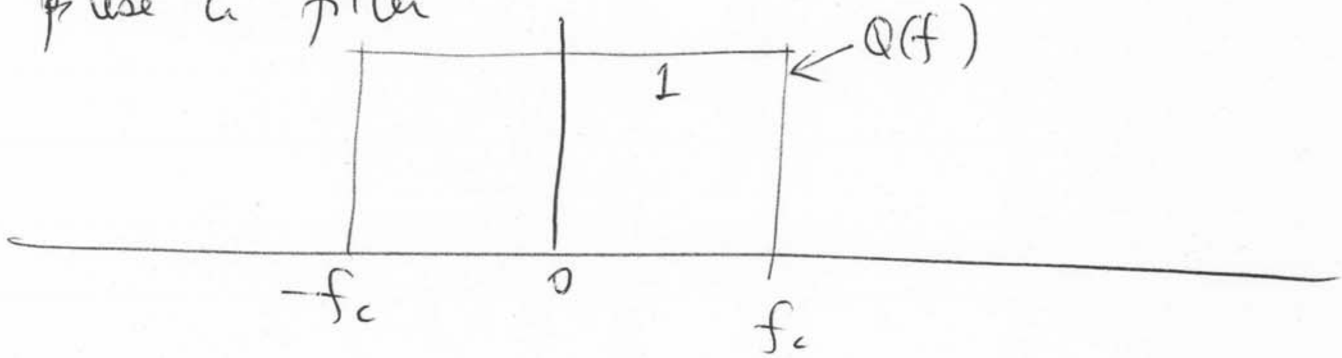
where $\text{sha}(f - \frac{1}{T}) = \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T})$







Now we are not interested in the images. So we use a filter



Thus we claim

$$\mathcal{F}(\hat{h}) \cdot Q(f) = \text{Fourier transform of the } h(t)$$

$$H \cdot F(h) = \mathcal{F}(\hat{h}) \cdot Q(f)$$

$$\Rightarrow h(t) = \mathcal{F}^{-1}\left(\mathcal{F}(\hat{h}) \cdot Q(f)\right)$$

$$\boxed{h(t) = h(nT) \sum_{-\infty}^{\infty} \delta(t - nT) * \frac{\sin[2\pi f_c(t - nT)]}{\pi(t - nT)}}$$

$$= \mathcal{F}$$

At $t = nT$ $\quad h(t) = \hat{h}(nT)$

elsewhere the above formula allows you to interpolate between samples.

BAND PASS SAMPLING THEOREM

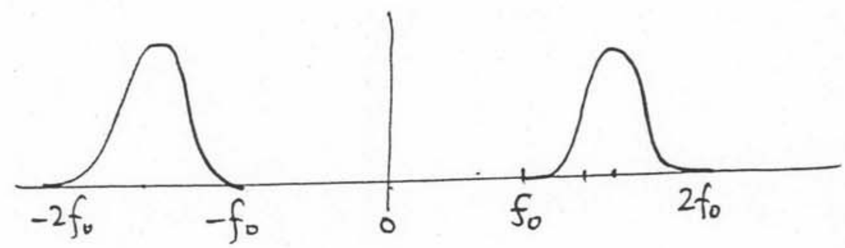
CRITICAL SAMPLING FREQUENCY

$$= \frac{2 f_h}{\text{largest integer not exceeding } \left(\frac{f_h}{f_h - f_l} \right)}$$

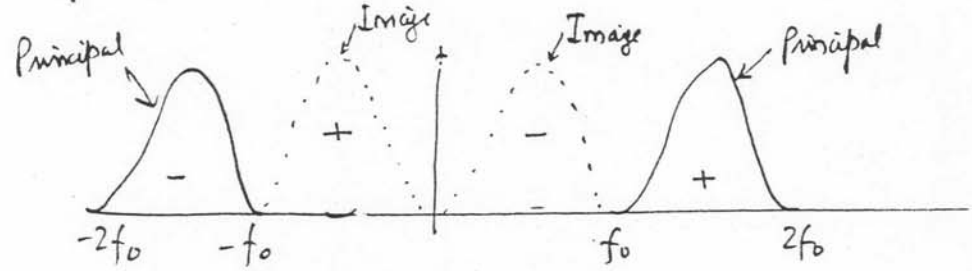
f_h = upper cutoff frequency
f_l = lower cutoff frequency.

This leads to clever application of sampling.

ex. Let signal ~~be~~ have power spectrum



Naively you will attempt sampling at $f_s = 4 f_0$. However sampling at $f_s = 2 f_0$ is sufficient.



The only inconvenience is that the positive & negative frequencies are swapped.

DISCRETE FOURIER TRANSFORMS

Let $h(t)$ be the signal. Then

$$y(t) = \left[h(t)\cdot\sum\delta(t-nT)\right]\cdot \text{window}(T_0)$$

↑ sampled signal — ↑ sampling — ↑ finite window

window(T_0) = finite samples

$$y(t) = \sum_{k=0}^{N-1} h(kT)\,\delta(t-kT)$$

Then

$$Y(f) = \sum h(kT)\,e^{-j2\pi fkT}$$

It is <u>customary</u> to ~~sam~~ evaluate or sample $Y(f)$ at $f = \frac{n}{NT}$ where NT is the total duration of the waveform

$$Y\left(\frac{n}{NT}\right) = \sum_{k=0}^{N-1} h(kT)\,e^{-j2\pi\frac{nk}{N}}$$

Why is the choice of $\Delta f = \frac{1}{NT}$ customary?

Several reasons.

- Since the Fourier transform repeats every $\frac{1}{T}$ we get only N distinct ~~compl~~ frequencies. Of this, due to Hermetean symmetry, only $N/2$ numbers are distanict. Thus the number of independent numbers is still N. which makes good physical sense.

Other choices of Δf lead to $> N$ or $< N$ frequencies.

Special points:

$$Y(0) = \sum_{k=0}^{N-1} h(kT) = \sum h_k$$

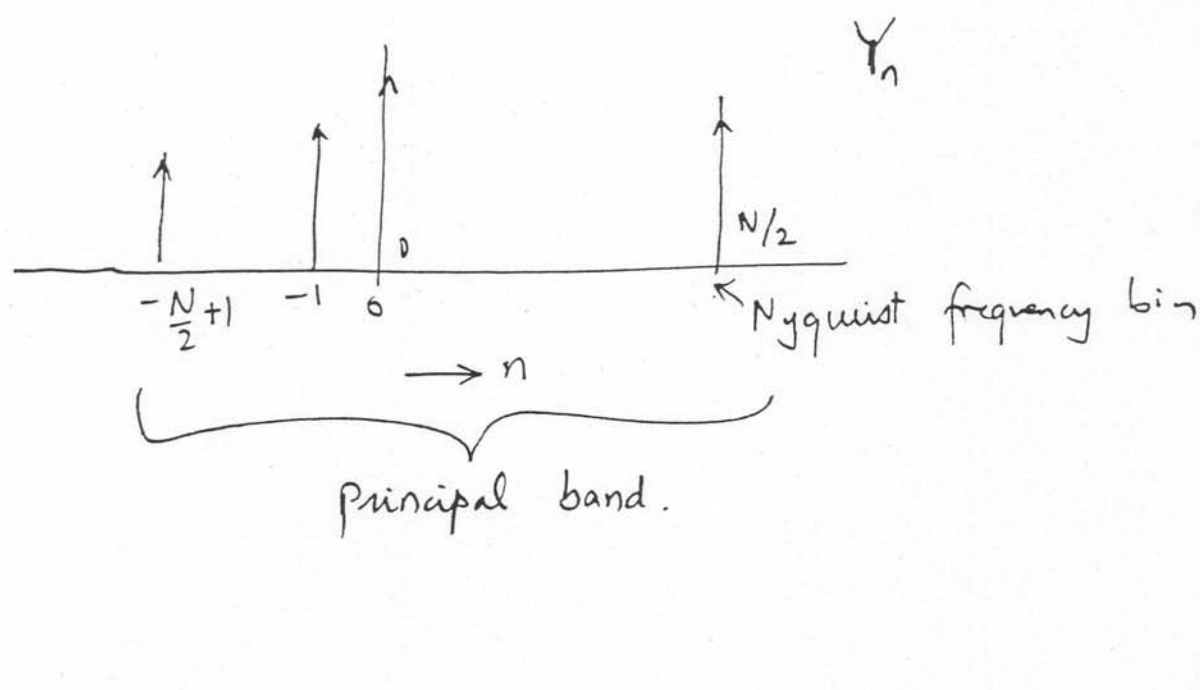
$$Y_0 = \sum_{k=0}^{N-1} h_k$$

Thus dc channel is purely real.

$$Y_{\frac{N}{2}} = \sum_{k=0}^{N-1} h_k e^{-j\pi k}$$

$$= (h_0 - h_1 + \cdots)$$

$$= \text{purely real.}$$



Discrete Fourier Transforms.

Most properties similar to continuous Fourier transforms except the following.

$h(k)$ & $H(n)$ are Fourier transform pairs then

$$h(kT) = \frac{1}{N} \sum_{n=0}^{N-1} H\left(\frac{n}{NT}\right) e^{j 2\pi nk/N}$$

$$H\left(\frac{n}{NT}\right) = \sum_{k=0}^{N-1} h(kT)\, e^{-j 2\pi nk/N}$$

Note the asymmetric "$1/N$" factor.

Parsevals Theorem:

$$\sum_{k=0}^{N-1} h^2(k) = \frac{1}{N} \sum_{k=0}^{N-1} |H(n)|^2$$

Note again the asymmetry.

DISCRETE FOURIER TRANSFORM

In real life you will have to take Fourier transform of set of discrete points i.e. a discrete Fourier transform (DFT). Two popular examples are (1) searching for pulsars and (2) obtaining the power spectrum of a signal from the measured autocorrelation function (this will be thoroughly covered in a later lecture). The sampling theorem makes it clear that the data must be sampled regularly and that is the only case we consider. As before, let Δt be the sampling interval and let $x(k)$ be the sample of the signal at the instant $k\Delta t$. Here we assume that x is a bandlimited signal (of bandwidth B). Let the total number of samples be N. Then the discrete transform of x is given by

$$X(l) \propto \sum_{k=0}^{N-1} x(k) \exp(-2\pi j l \Delta f k \Delta t)$$

where $j = \sqrt{-1}$ and the Fourier transform, $X(f)$ is evaluated at regularly spaced frequencies, $l\Delta f$, $l = 0, 1, ...$ Two questions arise:

- What should the value of Δf be i.e. what is the *spectral resolution* of the DFT? Clearly, smaller the Δf the better; otherwise you may be losing some information. On the other hand, there is the practical issue of having too small a Δf that the resulting array, $X(l\Delta f)$ becomes too unwieldy in size.

- How many steps in frequency should the DFT be calculated? Here we have some guidance from the sampling theorem (§4.1). There, we were specifically reminded that it is pointless to construct the Fourier transform for frequencies exceeding the Nyquist folding frequency $= f_s/2$ where f_s is the sampling rate. Thus the maximum index of the DFT is $M = \frac{1}{2} f_s/\Delta f$.

The first question is similar to the issue of the spectral resolution of gratings. (The analogy is exact; each ruling corresponds to one sample). There we learned that the spectral resolution of a grating is inversely related to the length or the total number of rulings. Likewise here the spectral resolution is related to the total length of time over which the signal was sampled, $N\Delta t$. The spectral resolution is approximately $1/N\Delta t$. Let us see why.

How does one define spectral resolution? If you had a spectrograph then you would look at an arc line i.e. a line with no (essentially) intrinsic width and then measure the full width at half maximum (FWHM) of this line as mapped by your spectrograph and call that FWHM as the spectral resolution. Likewise, here we will consider a sinusoidal signal of frequency f_0, $s(t)$ which we know has an intrinsic width that is zero. However, the sinusoid seen by our DFT is of finite length (duration $N\Delta t$). It is this truncation which leads to a finite spectral resolution. Our signal can then be modelled as $s(t) \cdot w(t)$ where $w(t)$ is the *window* function which is unity for $0 \leq t \leq T \equiv (N-1)\Delta t$ and zero otherwise,

$$x(k) = s(k) \cdot w(k).$$

By the convolution theorem, we know

$$X(l) = \mathcal{F}[s] * \mathcal{F}[w].$$

Since $\mathcal{F}[s] = \delta(f - f_0)$ and $\mathcal{F}[w] \propto \mathrm{sinc}(\pi f T)$ we have

$$X(l) \propto \mathrm{sinc}[\pi(l\Delta f - f_0)T].$$

i.e. $X(l)$ is centered at $f = f_0$ and has a FWHM of $1.2/T$ and thus our initial assertion is proved. For an arbitrary input signal, $x(t)$, the measured spectrum is the convolution of the true spectrum and the Fourier transform of the window function. This convolution defines the resolution of the measured spectrum.

As with optical spectrographs, it is wise to sample the impulse response function (i.e. in this case the response to a sinusoidal signal) at least as fine as two frequency samples per FWHM of the impulse response function. This criterion then suggests that $\Delta f = 0.6/T$ and thus $\Delta f \Delta t = 0.6/N$ Since $\Delta t = f_s^{-1}$ we note the maximum number of amplitudes that need to be estimated is

$$M = \frac{1}{2} f_s / \Delta f = \frac{N}{1.2}.$$

The Fast Fourier Transform

FFT is a particular implementation of the DFT. A brute force DFT requires $N \times M$ complex multiplications and additions. Since $M \sim N$ this becomes $\mathcal{O}(N^2)$. In contrast, an FFT algorithms takes $\mathcal{O}[\mathcal{N} \ln(\mathcal{N})]$.

The FFT algorithm, as normally implemented on most machines choose $\Delta f = 1/T$, cruder than one might ideally want. Higher resolution can be achieved by artificially increasing the time series (by padding the time series with zeros) and given the speed of an FFT, you are usually still better off using an FFT as opposed to a brute force DFT.

There are many FFT packages that are available on almost any computer system. The usual FFT package accepts a N element complex array $x(k) + jy(k)$ and does the equivalent summation and spew out $X(l) + jY(l)$,

$$X(l) + jY(l) = \sum_{k=0}^{N-1} [x(k) + jy(k)] \exp 2\pi jkl/N,$$

for $l = 0, 1, 2, ..., N/2$. Note that with the choice of Δf made by the FFT algorithm $\Delta f \Delta t = 1/N$ and that is why the exponential term looks particularly simple. (Some FFTs may normalize the above result by N but most packages evaluate the above sum, as shown above.)

For most of the cases, the input time series is real i.e. $y(k) = 0$. In that case, the term $l = 0$ corresponds to determining the d.c. or the sum of the signal. Thus it is purely real. The amplitude at the Nyquist folding frequency $(f_s/2)$ is

$$X(N/2) = \sum_{k=0}^{N-1} x(k)(-1)^k$$

is also purely real. Thus the ouput array consists of 2 reals, $(N - 2)/2$ complex numbers equalling a total of N separate components – equal to the number of samples put in.

Applying the DFT using an FFT algorithm

See Chapter 9 in the book on FFT by Brigham.

Leakage and Window Functions

The DFT of a sinusoid whose frequency f_0 is harmonically related to the sampling frequency is a δ function centered at the appropriate center frequency (see Figure). However, this is deceptive since we know the response of a DFT (as evaluated using an FFT algorithm) to a finite time series is a $\text{sinc}[\pi(f - f_0)T]$ and not a δ-function. What has happened is that the FFT evaluates the sinc-function at $f = l/T$, $l = 0, 1, ..., N/2$, the locations of the nulls of the sinc-function (apart from $f = f_0$). This results in the illusion that the response function is a δ-function.

In the more typical case where f_0 is not harmonically related to the sampling frequency the response of the FFT looks like a sinc function (see Figure). For example, assume that $f_0 = (L + 1/2)\Delta f$ in which case, significant power is seen at $l = L \pm 1, \pm 2, \pm 3...$ For example, the first side-lobe is about 21.7% of the peak value – not a particularly inspiring spectral response function.

This phenomenon is called as "leakage" since power from the frequency bin of interest is leaking into other bins. Leakage is a consequence of the finite length of the sample and the discontinuities introduced in the data due to this finite length (i.e. the Gibbs phenomenon). The latter aspect results in converting a band-limited signal into a band-unlimited data (a discontinuity generates frequencies all the way to infinity).

It is important to note that in either case, the *power* integrated over the response is exactly the same. For example, in the first case let us assume the input signal is $A\cos(2\pi f_0 t)$ in which case we know the power integrated over the frequency domain is $2 \times A^2/4N$ where the factor of 2 comes from equal contributions from positive and negative frequencies. In the second case, the integral power can be shown to be the same as in the previous case.

As in optical spectroscopy, leakage can be reduced by "apodizing" or "tapering". Tapering functions minimize leakage by suppressing the discontinuity in the data introduced by finite data length. There are quite a few choice of tapering functions (see below) but remember that the basic trade-off: *leakage reduction can be achieved only at the expense of broadening of the spectral response.*

Hanning. Hanning tapering consists of multiplying the input time series by the tapering function

$$w(t) = \frac{1}{2} - \frac{1}{2}\cos 2\pi t/T.$$

At the two end-points $t = 0$ and $t = T$ we note that $w(t)$ vanishes. The corresponding Fourier transform of the Hanning window is

$$W(f) = \frac{1}{2}Q(f) + \frac{1}{4}[Q(f + \Delta f) + Q(f - \Delta f)]$$

where $Q(f) = \text{sinc}(\pi T f)$. The FWHM of $W(f)$ is $1/T$ instead of $1.2/T$ as in the uniform window case. Thus the expected spectral broadening has happened. However, the level of the first side-lobe is now only 2.6% of the peak, an order of magniutude smaller than that of the uniform case.

Examples of other tapering functions are given in the attached table.

4.2.4

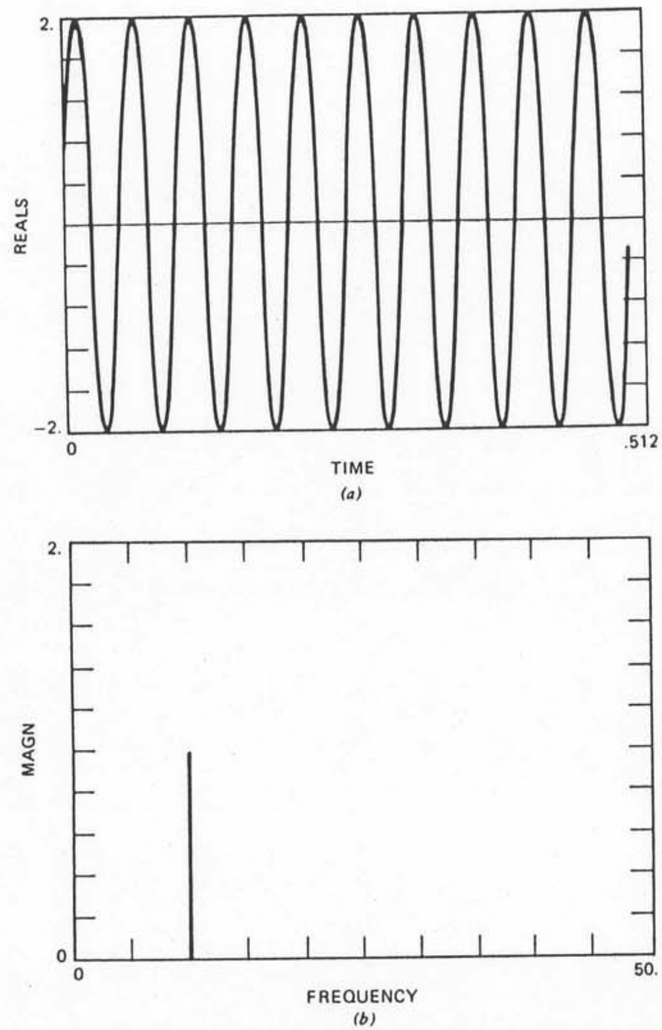


Figure 8.7 PSD of sine wave with integral number of cycles (10 cycles).

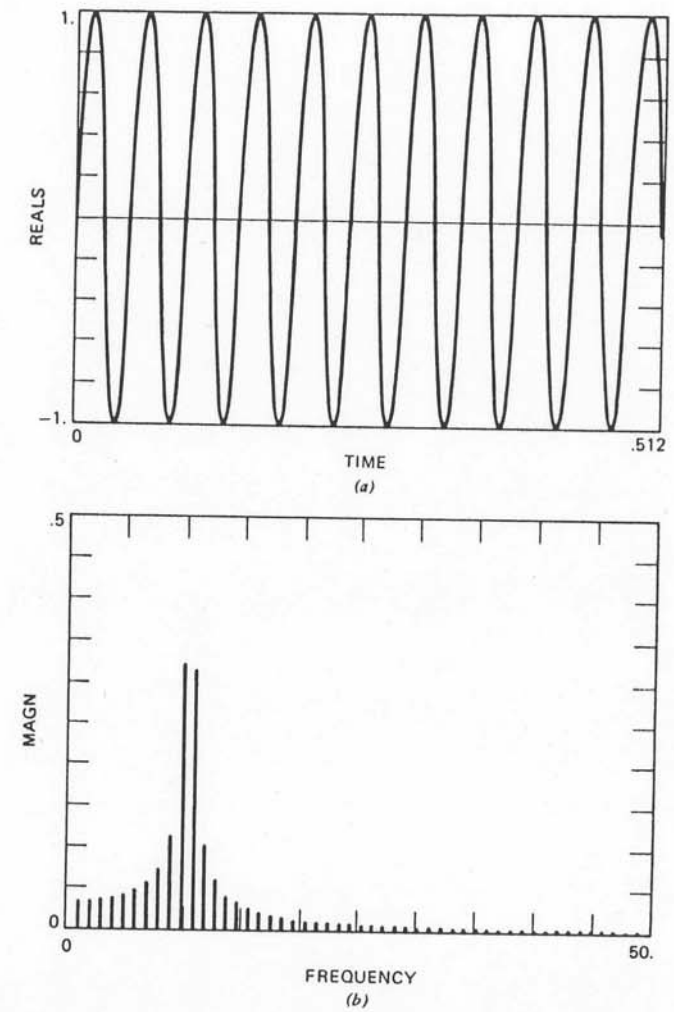


Figure 8.8 PSD of sine wave with integral number of cycles plus half cycle (10.5 cycles).

Assumes symmetrical window function between $-\tau_{max}$ & $+\tau_{max}$

[Earlier $w(t)$ was defined to be between $t=0$ to $t=t_m$]

TABLE II. Weighting Functions

Shape	Weighting function[a] $w(\tau)$	Scanning function $W(f)$	Scanning function width[b] δf	Noise bandwidth[c] B_N	$(\Delta f/B_N)^{1/2}$	Max. sidelobe level (%)
Uniform	1	$2\tau_{max}\,\text{sinc}\,2f\tau_{max}$[d]	$0.605/\tau_{max}$	$0.5/\tau_{max}$	1.099	21.7
Triangular	$1 - \lvert\tau\rvert/\tau_{max}$	$\tau_{max}\,\text{sinc}^2\,f\tau_{max}$[d]	$0.885/\tau_{max}$	$0.667/\tau_{max}$	1.16	4.7
Cos^2	$0.5(1 + \cos \pi\tau/\tau_{man})$	[e]	$1/\tau_{max}$	$0.75/\tau_{max}$	1.155	2.6
Cos^2 on pedestal	$\frac{1+a}{2} + \frac{1-a}{2}\frac{\cos \pi\tau}{\tau_{max}}$	[f]	$0.85/\tau_{max}$[g]	$0.635/\tau_{max}$[g]	1.155[g]	1.7[g]

[a] For $\lvert\tau\rvert \leqslant \tau_{max}$; $w(\tau) = 0$ elsewhere; $\tau_{max} = N\,\Delta t$.

[b] Measured at half-maximum of $W(f)$.

[c] $B_N = [1/W^2(0)] \int_{-\infty}^{\infty} W^2(f)\,df = [1/W^2(0)] \int_{-\infty}^{\infty} w^2(\tau)\,d\tau$.

[d] Here sinc $x = \sin x/x$.

[e] $W(f) = \tau_{max}[\text{sinc}(2f\tau)_{max}) + 4f\tau_{max} \sin(2\pi f\tau_{max})/2\pi(1 - 4f^2\tau^2{}_{max})]$.

[f] $W(f) = \tau_{max}[(1 + a\,\text{sinc}(2f\tau_{max}) + 4(1-a)f\tau_{max} \sin(2\pi f\tau_{max})/2\pi(1 - 4f^2\tau_{max}^2)]$.

[g] $a = 0.15$, approximate optimum value.

4.2.5