When is νf_ν useful?

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Abstract

Instruments usually measure the photon rate or power over a bandpass and from which we can extract the power per unit area per spectral unity or the spectral flux density, $f(\nu)$. However, what is of astrophysical interest is the spectral flux density integrated over frequency range which essentially captures most of the energy, $I(\nu_1, \nu_2)$. Frequently the quantity $\nu f(\nu)$ is used as a surrogate for I.

Here, I show that for power law spectra over the photon index range $1 < \alpha_P < 3$ (with $\alpha_P = 2$ for the Crab nebula over the X-ray and hard X-ray bands) the quantity $\log(\nu_2/\nu_1)\nu f(\nu)$ evaluated at $\nu_g = (\nu_1\nu_2)^{1/2}$, the geometric mean of the frequency bounds, is an excellent surrogate for I.

Finally, (perhaps an obvious point), in general νf_ν can be quite *misleading* when line emission is plotted along with continuum emission.

1 Spectral Flux Density

The spectral density is the energy received per unit time in a specific bandwidth. Spectral flux density is the energy received per unit area per unit time in a specific bandwidth.

Radio astronomers prefer frequency spectral flux density with the bandwidth set to 1 Hz. Optical astronomers prefer the wavelength spectral flux density in which the bandwidth is set to either 1 Å, 1 nm or 1 μm.

$$\begin{aligned} f(\nu) &= \frac{dF}{d\nu} \\ f(\lambda) &= \frac{dF}{d\lambda} \end{aligned} \qquad (1)$$

where F is the flux density (energy per unity time per unit area).

X-ray astronomers use keV and sometimes the photon flux density which is the number of photons per unit time per unit area usually in a frequency bandwidth of $\nu_0 = h/E_0$ where E_0 =1 keV and h is Planck's constant:

$$\frac{dN}{dE} = \frac{f(\nu)}{h\nu}\nu_0. \tag{2}$$

Power law spectra are often seen in astronomical sources. Using the radio astronomy convention,

$$f(\nu) = A_*(\nu/\nu_*)^{-\alpha} \tag{3}$$

where α is the spectral index (the negative sign being dropped by convention). The photon flux density is then $dN/dE \propto \nu^{-\alpha-1}$. Thus the "photon index", α_P, is one unit larger than α (the negative sign being understood). The X-ray spectrum of the Crab Nebula is a power law with $\alpha = 1$. The photon index is thus $\alpha_P = 2$.

2 The use of $\nu f(\nu)$

The numerical value $f(\nu)$ depends on value of the bandwidth which is arbitrary. 1 Hz could be much smaller than the range over which the object emits power (the usual case) or be too fine a range (say some sort of a cavity in resonance). Furthermore, $f(\lambda) = f(\nu)|d\nu/d\lambda| = f(\nu)\nu^2/c$ and thus has a different shape compared to $f(\nu)$. This difference in shape has led to the usual (not particularly meaningful) trick questions in elementary classes: at what frequency does the black body spectrum peak and why is this different from the peak wavelength?

The relationship

$$\mathcal{F}(\nu) \equiv \nu f(\nu) = \lambda f(\lambda) \equiv \mathcal{F}(\lambda) \tag{4}$$

makes it attractive to consider this quantity rather than $f(\nu)$ or $f(\lambda)$. Indeed, the peak frequency ($\nu_{\rm max}$) or peak wavelength of a black body peak is the same when $\mathcal{F}(\nu)$ is differentiated with respect to ν. In particular, the integral of the Planck black-body function (I_{BB}) is

$$\int_0^\infty I_{BB}(\nu)d\nu = \frac{\sigma T^4}{\pi} = 1.3586\,\mathcal{F}(\nu_{\rm max}). \tag{5}$$

Equation 5 says that the bolometric flux density is, within 36%, approximated by the maximum of $\mathcal{F}(\nu)$.

The two principal motivations to the use of $\mathcal{F}(\nu)$ are best summarized by Equations 4 (invariance of shape to the frequency spectral flux density or wavelength spectral flux density) and 5 (as a surrogate for the bolometric flux). See Gehrels[1] for a summary of the history of the usage of $\mathcal{F}(\nu)$ in astronomy. In that note, Gehrels points out two incorrect statements about $\mathcal{F}$: it is not the energy within an octave and neither within a decade. It is the flux within a *logade*[1] (a factor of of $e \sim 2.7$) of frequency:

$$\mathcal{F}(\nu) = \nu \frac{dF}{d\nu} = \frac{dF}{d(\log(\nu))}. \tag{6}$$

3 Power Law Spectrum

In this section we focus on the bolometric flux density for power law spectra. Integrating a power law spectrum (Equation 3) between ν_1 and ν_2 yields

$$I_\alpha(\nu_1, \nu_2) = A_* \nu_* \frac{1}{1-\alpha} \left[\left(\frac{\nu_2}{\nu_*} \right)^{-\alpha+1} - \left(\frac{\nu_1}{\nu_*} \right)^{-\alpha+1} \right]. \tag{7}$$

Evaluation of Equation 7 requires knowledge of $A_* \nu_*^\alpha$ and α.

For the special case of $\alpha = 1$,

$$\begin{aligned} I_1(\nu_1, \nu_2) &= A_* \nu_* \log(\nu_2/\nu_1) \\ &= f(\nu) \nu \log(\nu_2/\nu_1). \end{aligned} \tag{8}$$

The form of Equation 8 motivates us to define the following quantity as a surrogate for $I(\alpha)$:

$$\begin{aligned} F(\nu) &\equiv \log(\nu_2/\nu_1) \nu f(\nu) \\ &= \log(\nu_2/\nu_1) A_* \nu_* (\nu/\nu_*)^{-\alpha+1}. \end{aligned} \tag{9}$$

Given the definition of $F(\nu)$ I define the "bolometric" correction as

$$\begin{aligned} C_\alpha(\nu) &\equiv \frac{I_\alpha(\nu_1, \nu_2)}{F(\nu)} \\ &= \frac{1}{(1-\alpha)} \frac{1}{\log(\nu_2/\nu_1)} \Big[\Big(\frac{\nu_2}{\nu} \Big)^{-\alpha+1} - \Big(\frac{\nu_1}{\nu} \Big)^{-\alpha+1} \Big]. \end{aligned} \tag{10}$$

I evaluate the bolometric correction at the two bounding frequencies, ν_1 and ν_2 as well as the geometrical mean between the two frequencies, $\nu_g = (\nu_1 \nu_2)^{1/2}$:

$$C_\alpha(\nu_1) = \frac{1}{(1-\alpha)} \frac{1}{\log(\nu_2/\nu_1)} \Big[\Big(\frac{\nu_2}{\nu_1} \Big)^{-\alpha+1} - 1 \Big],$$

[1] This word is my invention.

$$C_\alpha(\nu_2) = \frac{1}{(1-\alpha)}\frac{1}{\log(\nu_2/\nu_1)}\Big[1 - \Big(\frac{\nu_1}{\nu_2}\Big)^{-\alpha+1}\Big],$$

$$C_\alpha(\nu_g) = \frac{1}{(1-\alpha)}\frac{1}{\log(\nu_2/\nu_1)}\Big[\Big(\frac{\nu_2}{\nu_1}\Big)^{\frac{-\alpha+1}{2}} - \Big(\frac{\nu_1}{\nu_1}\Big)^{\frac{-\alpha+1}{2}}\Big]. \qquad (11)$$

These correction factors are displayed in Figure 1.

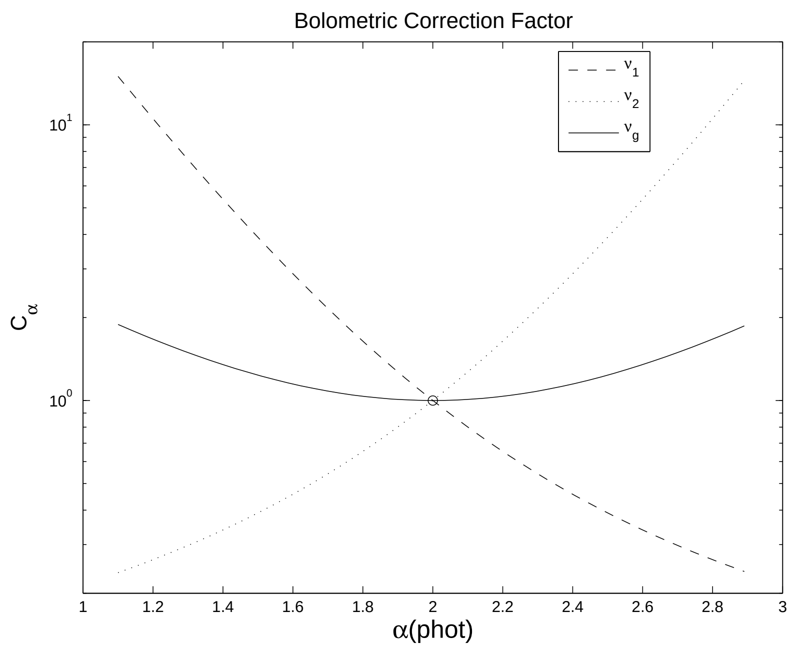

Figure 1: The bolometric factor, C_α, as a function of the photon index $(\alpha + 1)$. The frequency ratio, ν_2/ν_1 is set to 10^2.

Conclusion: The quantity

$$\nu_g F(\nu_g)\log(\nu_2/\nu_1) \qquad (12)$$

where ν_g is the geometric mean of ν_1 and ν_2 is an excellent surrogate for the flux integrated between ν_1 and ν_2.

Unfortunately, this (elegant) result as well as the result for the blackbody model (Equation 5) are what I call as parlor tricks. If you already have a black body fit then the bolometric flux is given by Stefan's formula . For the power law model, you need to have a very good sense of ν_1 and ν_2 in order to evaluate a meaningful ν_g (and have the security that $f(\nu_g)$ is measured or can be reasonably inferred). If you already had that assurance then you may as well use the exact formula (Equation 7).

The lesson is primarily pedagogical: $\mathcal{F}(\nu)$ can, with some caution and proper choice of frequency, be a good surrogate for the bolometric flux at least for two very different types of intensity distributions (black body and power law).

4 Line Emission

Broad-band emission from the Galactic Center region is displayed in Figure 2. The e^+-e^- line seemingly has the same flux as the continuum emission at higher energies. However, the line is very narrow (primarily arising from annihilation in the interstellar medium. In this case, νf_ν is an over-estimate by the factor $\nu/\Delta\nu$ where $\Delta\nu$ is the width of the line.

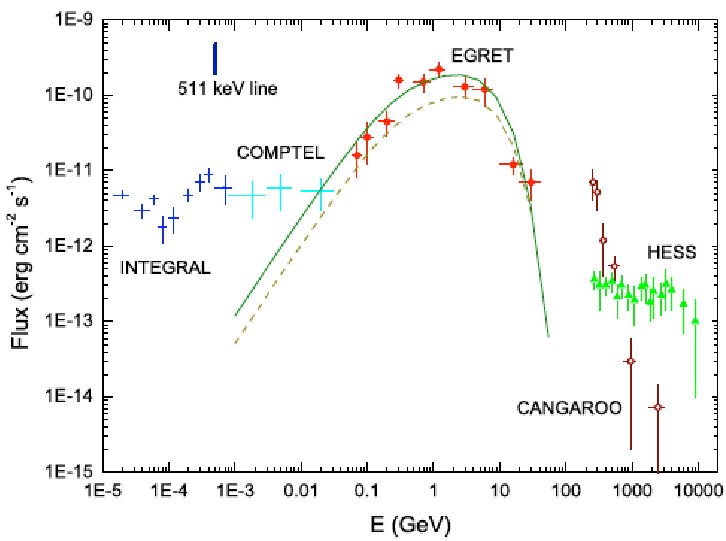

Figure 2: Broad band emission from the Galactic Center

5 Photon index and spectral index

In X-ray and γ-ray astronomy it is practical to measure the photon flux i.e. the number of photons per square centimeter per second. This is usually denoted by $\mathcal{N}(E)$. The spectral photon flux is the number of photons per unit frequency bandwidth per unit area per second. It makes little sense to use Hz as the unit bandwidth. A practical unit is keV (X-ray) or MeV (γ ray).

$$\mathcal{N}(E): \qquad \text{photons}\,\text{cm}^{-2}\,\text{s}^{-1} \tag{13}$$

$$N(E): \qquad \text{photons}\,\text{cm}^{-2}\,\text{s}^{-1}\,\text{keV}^{-1}. \tag{14}$$

Thus, $EN(E)$ is the spectral energy flux and is $\propto f_\nu$ and $E^2N(E) \propto \nu f_\nu$.

A The Crab Nebula

Traditionally, the Crab nebula is used as a calibrator for classical (2–10 keV) and hard X-ray band (10–100 keV). According to Toor & Seward [2]

$$I = AE^\alpha \exp(-\sigma N_H) \tag{15}$$

where $A = 9.7\,\text{keV}\,\text{keV}^{-1}\,\text{cm}^{-2}\,\text{s}^{-1}$, $\alpha = 1.1 \pm 0.03$ and the exponential term represents the ISM absorption.

Thus $I(1\,\text{keV}) = A$ or $I = 9.4$ photon $\text{cm}^{-2}\,\text{s}^{-1}\,\text{keV}^{-1}$. The Jansky unit is spectral density but per Hz and so we find $I(\nu = 2.4 \times 10^{17}\,\text{Hz}) = 6.4 \times 10^{-26}$ $\text{erg}\,\text{cm}^{-2}\,\text{s}^{-1}\,\text{Hz}^{-1}$. Thus at $E = 6\,\text{keV}$ the flux is about 1 mJy whence the usual statement "1 Crab = 1 milliJy".

Ignoring the exponential term, the flux between photon energy E_1 and E_2 (both in keV), given that α is almost 1, is

$$F = A\log(E_2/E_1)\,\text{keV}\,\text{cm}^{-2}\,\text{s}^{-1}. \tag{16}$$

Since 1 keV=1.6×10^{-9} erg we find

$$\begin{aligned} F(2-10\,\text{keV}) &= 2.5 \times 10^{-8}\,\text{erg}\,\text{cm}^{-2}\,\text{s}^{-1} \\ F(20-50\,\text{keV}) &= 1.4 \times 10^{-8}\,\text{erg}\,\text{cm}^{-2}\,\text{s}^{-1}. \end{aligned} \tag{17}$$

[1] N. Gehrels, Ill Nuovo Cimento B 112, pp 11 (1997)
[2] Toor & F. D. Seward, ApJ 79, pp 995 (1974)