$J_\nu(t)$ ‡ is the incident flux at time t, $\nu_o(x)$ is the ionization edge for the ion X^{+n}, $a_\nu(x)$ is the frequency-dependent photoionization cross section for the ion X^{+n}, and α_R is the recombination rate for recombinations from $X^{+(n+1)}$ to X^{+n}, *i.e.* $\alpha_R(X^{+n})$. Let Γ^* and x^* be the equilibrium values at t=0, and let $\tau = t n_e \alpha_R$. Let $\gamma(\tau) = \Gamma(\tau)/\Gamma^*$, and we have:

$$\frac{dx}{d\tau} = (1-x) - x\gamma(\tau)\left[\frac{\Gamma^*}{n_e \alpha_R}\right] \quad .$$

In equilibrium, at τ=0, we have: $\alpha_R n_e(1-x^*) = \Gamma^* x^*$, with $\gamma(0)$=1. So we define: $k \equiv (1-x^*)/x^* = \Gamma^*/n_e\alpha_R$. We further define: $f(\tau) \equiv 1 + \gamma(\tau)k$, so we have:

$$\frac{dx}{d\tau} + x f(\tau) = 1 \quad .$$

This is a "linear differential equation of the first order", and can be solved by the use of an integrating factor (*cf.* Rainville 1989). Let $\omega(\tau) = \int_0^\tau f(\tau')d\tau'$, such that $d\omega(\tau)/d\tau = f(\tau)$. Then:

$$\frac{d}{d\tau}[e^{\omega(\tau)}x(\tau)] = e^{\omega(\tau)} \quad .$$

Integrating both sides from 0 to τ, and applying our initial condition, we have:

$$x(\tau) = e^{-\omega(\tau)}\left(x(0) + \int_0^\tau e^{\omega(\tau')}d\tau'\right) \quad .$$

We can now determine the fractional abundance of the X^{+n} ion, x, as a function of time given an initial fractional abundance and a relative light curve $\gamma(\tau)$, such that $\gamma(0) = 1$. Note that the fractional abundance calculated by this solution will be correct only for this idealized case, neglecting thermal changes and other effects which were included within CLOUDY to derive our equilibrium fractional abundances.

To get a more physical sense of our solution, let us assume that the photon flux changes instantaneously at t=0 to a constant level $\gamma(\tau > 0) = c$. We then have: $\omega(\tau) = \tau(1+ck)$, and the solution is:

$$x(\tau) = \left(x(0) - \frac{1}{1+ck}\right)e^{-\tau(1+ck)} + \frac{1}{1+ck} \quad .$$

‡ $4\pi J_\nu \equiv L_\nu/4\pi r^2$.