

$$\begin{aligned}
\frac{dx_0}{dt} &= R_0 x_1 - P_0 x_0 \\
\frac{dx_1}{dt} &= R_1 x_2 - P_1 x_1 + P_0 x_0 - R_0 x_1 \\
&\vdots \\
\frac{dx_i}{dt} &= R_i x_{i+1} - P_i x_i + P_{i-1} x_{i-1} - R_{i-1} x_i \\
&\vdots \\
\frac{dx_z}{dt} &= P_{z-1} x_{z-1} - R_{z-1} x_z \\
\sum_{i=0}^z x_i &= 1 \quad .
\end{aligned}$$

The interdependence of the equations makes an analytic solution difficult, however, we can use numerical methods to get solutions for the ions and initial conditions of interest. We can use a “Runge-Kutta” method for a rather straightforward solution (see, for example, Press *et al.* 1992). This basically involves integrating the right hand side of the above equations from time t to $t + \delta t$, assuming a constant integrand during the small time interval δt .

However, before applying numerical techniques directly we must solve for the equilibrium initial conditions to find the values for Γ_i^* . Unfortunately, we do not know the initial equilibrium fractional abundances appropriate for our idealized equations. If we use the equilibrium values derived by CLOUDY for the multiple-ion case, the time-dependent solution will be unstable. Although, they may have been adequate for qualitative estimates in the two-ion case, the CLOUDY values do not satisfy the equilibrium conditions $dx_i/dt=0$ at $t=0$, for our simplified equations. In order to determine the recombination delay function, we must first solve the idealized equilibrium equations using exactly the same parameters which we use to solve the time-dependent equations.

In equilibrium, the above equations simplify considerably. For any ion i , we have:

$$0 = R_i x_{i+1} - P_i x_i \quad ,$$